The infinite unitary group, Howe dual pairs, and the quantization of constrained systems

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Abstract

The irreducible unitary representations of the Banach Lie group $U_0(\mathcal{H})$ (which is the norm-closure of the inductive limit $\cup_k U(k)$) of unitary operators on a separable Hilbert space $\mathcal{H}$, which were found by Kirillov and Ol'shanskii, are reconstructed from quantization theory. Firstly, the coadjoint orbits of this group are realized as Marsden-Weinstein symplectic quotients in the setting of dual pairs. Secondly, these quotients are quantized on the basis of the author's earlier proposal to quantize a more general symplectic reduction procedure by means of Rieffel induction (a technique in the theory of operator algebras). As a warmup, the simplest such orbit, the projective Hilbert space, is first quantized using geometric quantization, and then again with Rieffel induction.

Reduction and induction have to be performed with either $U(M)$ or $U(M,N)$. The former case is straightforward, unless the half-form correction to the (geometric) quantization of the unconstrained system is applied. The latter case, in which one induces from holomorphic discrete series representations, is problematic. For finite-dimensional $\mathcal{H} = \mathbb{C}^k$, the desired result is only obtained if one ignores half-forms, and induces from a representation, 'half' of whose highest weight is shifted by k (relative to the naive orbit correspondence). This presumably poses a problem for any theory of quantizing constrained systems.

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1 Introduction

1.1 Marsden-Weinstein reduction and constrained systems

The reduced phase space of a constrained mechanical system [15] may often be written as a so-called Marsden-Weinstein quotient [36, 1, 17] of the phase space of the unconstrained system. Mathematically, this means that certain complicated symplectic manifolds can be constructed from perhaps less complicated ones using a canonical reduction procedure [39].

For example, the complex projective space $\mathbb{C}P^n$ (equipped with the usual Kähler structure [16]) is a Marsden-Weinstein quotient of $\mathbb{C}^{n+1}$ (whose symplectic form ω is expressed in terms of the standard inner product, taken linear in the first entry, by $\omega(\psi, \varphi) = -2\,\mathrm{Im}\,(\psi, \varphi))$ with respect to the group $U(1)$ [1]. Namely, $U(1)$ (identified with the unit circle in the complex plane) acts on $\mathbb{C}^{n+1}$ as follows: $\exp(i\alpha) \in U(1)$ maps $\psi \in \mathbb{C}^{n+1}$ to $\exp(i\alpha)\psi$; this action is symplectic, and yields an equivariant moment map [1] $J : \mathbb{C}^{n+1} \to \mathbf{u(1)}^* \equiv \mathbb{R}$ given by $J(\psi) = (\psi, \psi)$. Then $\mathbb{C}P^n \simeq J^{-1}(1)/U(1)$.

More generally, given a suitable symplectic action of a Lie group H on a symplectic space S one may construct a moment map [1, 17] $J : S \to \mathbf{h}^*$ (where $\mathbf{h}^*$ is the topological dual of the Lie algebra $\mathbf{h}$ of H). If J intertwines the H-action on S with the co-adjoint action on $\mathbf{h}^*$, the Marsden-Weinstein reduced space at $\mu \in \mathbf{h}^*$ is $S^\mu = J^{-1}(\mu)/H_\mu$, where H_μ is the stability group of μ under the coadjoint action [1, 17]. (If $\mathcal{O}_\mu$ is the co-adjoint orbit through μ one finds that $S^\mu \simeq J^{-1}(\mathcal{O}_\mu)/H$, so that the reduced space only depends on the orbit $\mathcal{O}_\mu$.) The reduced space (which is a manifold only under further assumptions) inherits its symplectic structure from S, and this may well be the most efficient way of defining the symplectic structure of certain spaces (the example above being a case in point). *Here and in what follows, actions and representations are assumed continuous.*

Marsden-Weinstein reduction is a special case of a more general symplectic reduction procedure [37, 60, 30]. It was recently proposed [30] that this more general procedure should be quantized by a technique from operator algebra theory known

as Rieffel induction [47, 13]. This proposal entails a new approach to the quantization of constrained mechanical systems, which so far has been succesfully tested in the theory of particles moving in external gravitational and Yang-Mills fields [30], abelian gauge field theories [31], and in a comparison with geometric quantization [48]. The purpose of the present paper is to provide further examples of this approach in the context of the quantization of infinite-dimensional Kähler manifolds.

1.2 Quantum mechanics

One motivation for choosing this class of examples comes from an intriguing observation of Tuynman [52, 53] to the effect that the quantization of quantum mechanics is quantum mechanics itself. Namely, the space of pure states of a quantum-mechanical system without superselection rules is the projective Hilbert space $\mathbb{P}\mathcal{H}$, which as a symplectic manifold may be regarded as the phase space of a classical system [34, 1]. The geometric quantization of this phase space then reproduces the original Hilbert space $\mathcal{H}$. We will review this argument in some detail in section 2, and complete it by considering the quantization of the observables. The key point is that it is the class of observables and their associated algebraic structure which distinguishes quantum mechanics from a possible classical theory defined on $\mathbb{P}\mathcal{H}$.

In ordinary quantum mechanics, any self-adjoint operator A (assumed bounded for simplicity) on $\mathcal{H}$ corresponds to an observable. Equivalently, one may define a real-valued function f_A on $\mathbb{P}\mathcal{H}$ by $f_A([\psi]) = (A\psi, \psi)$, where the unit vector $\psi \in \mathcal{H}$ is any lift of $[\psi] \in \mathbb{P}\mathcal{H}$. We shall find that these f_A are precisely the functions that are quantizable (in the sense of geometric quantization) in the holomorphic (or anti-holomorphic) polarization of $\mathbb{P}\mathcal{H}$. Other, more physical characterizations of these observables in the context of quantum mechanics on $\mathbb{P}\mathcal{H}$ will be given in subsection 2.4.

As mentioned above, the Kähler manifold $\mathbb{P}\mathcal{H}$ may be realized as a Marsden-Weinstein quotient, and as such it can be quantized by the Rieffel induction technique. Since this technique provides an alternative to geometric quantization (given a quantization of the unconstrained system $\mathcal{H}$), it is interesting to see how the spe-

cial role played by the quantum-mechanical observables comes about in the former approach.

1.3 Howe dual pairs and the representation theory of the unitary group

Moreover, we will apply our techniques to quantize a whole class of Kähler manifolds, namely the collection of quantizable co-adjoint orbits of the unitary group $U_0(\mathcal{H})$, which consists of all unitary operators U on $\mathcal{H}$ for which $U - \mathbb{I}$ is compact, and which carries the uniform topology; clearly $U(\mathbb{C}^k) = U(k)$ for $k < \infty$, and $U_0(\mathcal{H})$ is the norm-closure of the inductive limit $U(\infty) \equiv \cup_k U(k)$. $\mathbb{P}\mathcal{H}$ is one such orbit, and its usual Kähler symplectic structure coincides with the Lie-Kirillov symplectic structure of this orbit.) This application was motivated by Montgomery's observation [38] (also cf. [33]) that for finite-dimensional $\mathcal{H}$ some these orbits (namely those characterized by a collection of positive eigenvalues) are Marsden-Weinstein quotients of $\mathcal{H} \otimes \mathbb{C}^M$ with respect to $U(M)$ for suitable M (which depends on the orbit). We extend this result (which is a special instance of the theory of classical dual pairs [24, 50, 57]) to the situation where the eigenvalues may be of either sign, and also to the case where $\mathcal{H}$ is infinite-dimensional. In the general case one reduces with respect to the group $U(M, N)$.

Apart from its obvious relevance to quantum mechanics, our special interest in the infinite-dimensional separable case was triggered by the Kirillov-Ol'shanskii classification of all continuous representations of the Banach Lie group $U_0(\mathcal{H})$ [26, 40]. Note that the Fréchet Lie group $U(\mathcal{H})$ (consisting of all unitary operators on $\mathcal{H}$), equipped with the strong operator topology, has the same representation theory as $U_0(\mathcal{H})$, because all representations of $U_0(\mathcal{H})$ are also strongly continuous, and can therefore be extended to $U(\mathcal{H})$. Moreover, $U(\mathcal{H})$ retopologized with the uniform topology has the same irreducible representations on separable Hilbert spaces as the same group equipped with the strong topology (whose irreducible representation spaces are automatically separable). (The representation theory of $U(\infty)$ equipped with the inductive limit topology is much more complicated [43, 9] and will not be

discussed here.)

A remarkable aspect of this classification (and also the way it was found) is that all irreducible unitary representations of $U_0(\mathcal{H})$ may be thought of as the geometric quantization of certain of its coadjoint orbits. However, only the geometric quantization of orbits with positive eigenvalues may actually be found in the literature [8]; even this special case is already fairly involved (cf. the Borel-Weil theory (e.g., [5]) for finite-dimensional $\mathcal{H}$). It is this quantization that we will be able to redo, and much simplify, by regarding the orbit as a constrained system. With our formalism we merely have to quantize $\mathcal{H} \otimes \mathbb{C}^M$, which at first sight is rather trivially done by Fock space techniques, and apply Rieffel induction. This last step is easily carried out on the basis of Weyl's classical results on tensor products and the symmetric group [58, 20]. If, however, one uses the refined version of geometric quantization that incorporates half-forms [59] (leading to corrections that are only finite if $\mathcal{H} = \mathbb{C}^k$, $k < \infty$), then quantization and reduction fail to commute, and our method breaks down.

The general case (where the orbit is characterized by eigenvalues of arbitrary sign) is considerably more complicated than the special case of fixed sign. The 'answer' is known, in that it is clear from Kirillov's work [26] which representation of $U_0(\mathcal{H})$ (on a specific Hilbert space) forms the quantization of a given quantizable orbit, regarded as the reduced phase space. Also, the quantization of the unconstrained system $S = \mathcal{H} \otimes \mathbb{C}^{M+N}$ (with a specific symplectic structure depending on M and N) is known explicitly at least for finite-dimensional $\mathcal{H} = \mathbb{C}^k$: it is the k-fold tensor product of the metaplectic (alternatively called 'oscillator' or 'Segal-Shale-Weil') representation [14], restricted from $Sp(2(N+M), \mathbb{R})$ to its subgroup $U(M, N)$ (see [51, 50, 6]). This tensor product has been decomposed by Kashiwara and Vergne [25], also cf. [18]. (As in the compact case, one has the choice whether or not to incorporate half-forms.)

The best point of view concerning this quantization is provided by the formalism of Howe dual pairs [21, 18, 19], which, as already remarked in [24, 50, 57] (and to some extent anticipated in [51]), neatly emerges as the quantization of the theory

of classical (Weinstein) dual pairs [57] coming from symplectic group actions on a vector space. (A Howe dual pair is defined as a pair of reductive subgroups of a symplectic group $Sp(2n, \mathbb{R})$ which are each other's centralizer.) The trouble is that the decomposition of the Hilbert space quantizing $S = \mathbb{C}^k \otimes \mathbb{C}^{M+N}$ under $U(k)$ and $U(M, N)$ (which form a Howe dual pair) does not reflect the decomposition of S under these group actions if $k > M + N$ (which is the case of relevance to us, as we are eventually interested in $k = \infty$), cf. [3]. However, a certain modification of our method will lead to some success.

1.4 Rieffel induction for group actions

Let us close this Introduction by briefly reviewing how Rieffel induction [47, 13] (in the version of [30]) specializes to the present context. As we have mentioned, Rieffel induction quantizes a much more general symplectic reduction procedure than that of Marsden-Weinstein; the specialization of this technique to the quantization of Marsden-Weinstein reduction is only a slight generalization of the well-known Mackey induction technique for groups. The situation is further simplified if one deals with reductions by actions of compact groups (namely $U(M)$).

It is convenient to start from a *right* symplectic action of a connected Lie group H on S, so that the accompanying moment map $J : S \to (\mathbf{h}^*)$ is an anti-Poisson homomorphism w.r.t. the Lie-Kirillov Poisson structure on $\mathbf{h}^*$ [1, 17] (the latter is most easily defined in terms of the linear functions on $\mathbf{h}^*$; each $X \in \mathbf{h}$ defines such a linear function by evaluation, and the Poisson bracket on $C^\infty(\mathbf{h}^*)$ is then determined by $\{X, Y\} = [X, Y]$); we indicate this by writing $J : S \to (\mathbf{h}^*)^-$. We assume that the reduced space $S^\mu \equiv J^{-1}(\mathcal{O}_\mu)/H$ is a manifold.

We adhere to the point of view that symplectic spaces are best seen as modules for Poisson algebras, and regard the symplectic reduction procedure as a construction in the representation theory of Poisson algebras [29]. Thus we suppose that a Poisson subalgebra A of $C^\infty(S)$ is given, whose 'induced' representation π^μ on S^μ we wish to construct. A sufficient condition on A allowing this construction is that each element of A is H-invariant; given that H is connected, this may be reformulated algebraically

by requiring that A lie in the Poisson commutant of $J^*(C^\infty(\mathbf{h}^*))$. (A necessary and sufficient condition is that each element of A is H-invariant on $J^{-1}(\mathcal{O}_\mu)$.) The Poison algebra homomorphism $\pi^\mu : \mathsf{A} \rightarrow C^\infty(S^\mu)$ is then simply defined by the condition that $pr^*\pi^\mu(f) = f$ on $J^{-1}(\mathcal{O}_\mu)$ (here $pr : J^{-1}(\mathcal{O}_\mu) \rightarrow J^{-1}(\mathcal{O}_\mu)/H$ is the canonical projection). For example, if a Lie group G acts symplectically on S in such a way that its action commutes with the H-action, one could take $\mathsf{A} = J_L^*(C^\infty(\mathbf{g}^*))$ (where $J_L : S \rightarrow \mathbf{g}^*$ is a moment map, not necessarily equivariant, corresponding to the G-action).

Alternatively, one could forget the Poisson algebra A and simply regard S^μ as a symplectic G-space in the obvious way; one has then constructed an 'induced symplectic realization', or 'classical representation', of G itself, rather than of its associated Poisson algebra. The well-known symplectic induction procedure [24, 17] is a special case of this construction (it is obtained by taking $H \subset G$ and $S = T^*G$). Thus induction and reduction are the same; the former terminology is more appropriate when starting from $\mathcal{O}_\mu$, whereas the latter is natural when one has S in mind. In the main text we will take $S = \mathcal{H} \otimes \mathbb{C}^{M+N}$, $G = U_0(\mathcal{H})$, and $H = U(M, N)$ with their natural left- and right actions on S, respectively.

To quantize the reduced space S^μ and the associated induced representation of A or G, we assume that a quantization of the unconstrained system as well as of the constraints are given. In the examples studied in this paper, the required data specified below are obvious, and therefore we will refrain from giving an exact definition of 'quantization'; the term will be used in a somewhat loose way, and everyone's favourite definition will lead to the objects we use in our examples.

Hence we suppose we have firstly found a Hilbert space $\mathcal{F}$, which may be thought of as the (geometric) quantization of S (if $S = \mathcal{H} \otimes \mathbb{C}^M$ we take $\mathcal{F}$ to be the symmetric Fock space $\exp(S)$ over S). Secondly, a unitary right-action (i.e., anti-representation) π_R on $\mathcal{F}$ should be given, which is the quantization of the symplectic right-action of H on S (for $\mathcal{F} = \exp(S)$ this will be the second quantization of the action on S). Thirdly, we require a unitary representation $\pi_\chi(H)$ on a Hilbert space $\mathcal{H}_\chi$, which 'quantizes' the coadjoint action of H on the coadjoint orbit $\mathcal{O}_\mu$ This

is only possible if the orbit is 'quantizable'; for $H = U(M)$ there is a bijective correspondence between such orbits and unitary representations, and for $U(M,N)$ one obtains at least all unitary highest weight modules by 'quantizing' such orbits [2, 54]. (In the latter case the concept of quantization has to be stretched somewhat to incorporate the derived functor technique to construct representations.)

First assuming that H is compact, we construct the induced space $\mathcal{H}^\chi$ from these data as the subspace of $\mathcal{F} \otimes \mathcal{H}_\chi$ on which $\pi_R^{-1} \otimes \pi_\chi$ acts trivially (here π_R^{-1} is the representation of H defined by $\pi_R^{-1}(h) = \pi_R(h^{-1})$). If H is only locally compact (and assumed unimodular for simplicity) with Haar measure dh, one has to find a dense subspace $L \subset \mathcal{F}$ such that the integral $\int_H dh\, ((\pi_R^{-1} \otimes \pi_\chi)(h)\Psi, \Phi) \equiv (\Psi, \Phi)_0$ is finite for all $\Psi, \Phi \in L \otimes \mathcal{H}_\chi$. This defines a sesquilinear form $(\cdot,\cdot)_0$ on $L \otimes \mathcal{H}_\chi$ which can be shown to be positive semi-definite under suitable conditions [30]. The induced space $\mathcal{H}^\chi$ is then defined as the completion of the quotient of $L \otimes \mathcal{H}_\chi$ by the null space of $(\cdot,\cdot)_0$; its inner product is, of course, given by the quotient of $(\cdot,\cdot)_0$. For H compact the integral exists for all $\Psi, \Phi \in \mathcal{F}$ and $(\Psi, \Phi)_0 = (P_0\Psi, P_0\Phi)$, where P_0 is the projector onto the subspace of $\mathcal{F} \otimes \mathcal{H}_\chi$ carrying the trivial representation of H, so that we recover the first description of $\mathcal{H}^\chi$. (Even the case where H is not locally compact can sometimes be handled by a limiting procedure, cf. [31].)

We now assume that a group G or a *-algebra $\mathfrak{A}$ acts on $\mathcal{F}$ through a unitary representation or a *-representation (which we both denote by π_L), respectively; it is required that these actions commute with $\pi_R(H)$. The self-adjoint part of the *-algebra $\mathfrak{A}$ is thought of as the (deformation) quantization of the Poisson algebra A, and the actions of $\mathfrak{A}$ or G on $\mathcal{F}$ should be the quantum counterparts of the actions of A or G on S. (In our example, the action of $G = U_0(\mathcal{H})$ on $\mathcal{F} = \exp(\mathcal{H} \otimes \mathbb{C}^k)$ is the second quantization of the left-action of $U_0(\mathcal{H})$ on $S = \mathcal{H} \otimes \mathbb{C}^k$).

The induced representations $\pi^\chi(\mathfrak{A})$ or $\pi^\chi(G)$ on $\mathcal{H}^\chi$ are now defined as follows. For H compact, π^χ is simply the restriction of $\pi_L \otimes \mathbb{I}$ to $\mathcal{H}^\chi \subset \mathcal{F} \otimes \mathcal{H}_\chi$; this is well defined because $\pi_L \otimes \mathbb{I}$ commutes with $\pi_R^{-1} \otimes \pi_\chi$. In the general case, one has to assume that π_L leaves L stable; then π^χ is essentially defined as the quotient of the action of $\pi_L \otimes \mathbb{I}$ (on $L \otimes \mathcal{H}_\chi$) to $\mathcal{H}^\chi$ as defined above (cf. [30] for technical

details pertinent to the general case). The Mackey induction procedure for group representations is recovered by assuming that $H \subset G$, and taking $\mathcal{F} = L^2(G)$, cf. [47, 13] for details in the original setting of Rieffel induction, and [30] for the above setting.

As a simple example, take $\mathcal{F} = L^2(G)$ for a locally compact but non-compact unimodular group G, and $H = G$, which act on $\mathcal{F}$ in the left- and right-regular representations, respectively. We induce from the trivial representation $\pi_\chi = \pi_{\mathrm{id}}$. We may choose $L = C_c(G)$, and define $V : L \to \mathbb{C}$ by $V\psi = \int_G dx\, \psi(x)$. This integral exists, and $(V\psi, V\varphi) = (\psi, \varphi)_0$ (where the left-hand side is the inner product in $\mathbb{C}$). Hence we can identify the null space of $(\cdot, \cdot)_0$ with the kernel of V, and the induced space $\mathcal{H}^{\mathrm{id}}$ with the image of V, that is, with $\mathbb{C}$. The induced representation $\pi^{\mathrm{id}}(G)$ comes out to be the trivial one. This example illustrates the interesting point that Rieffel induction does not necessarily produce representations that are weakly contained in $\mathcal{F} \otimes \mathcal{H}_\chi$. For $G = \mathbb{R}^n$ it so happens that the trivial representation is weakly contained in the regular one, but for G semi-simple (and non-compact) it is not. Yet Rieffel induction manages to extract it in either case.

2 The geometric quantization of quantum mechanics

In this section we review (and somewhat elaborate on) Tuynman's argument that the geometric quantization of the symplectic formulation of quantum mechanics reproduces the usual Hilbert space formalism [52, 53], and complete the thesis by incorporating the quantization of the observables. We will keep the discussion technically simple by assuming that $\mathcal{H}$ is finite-dimensional (the infinite-dimensional case will be dealt with later, using the appropriate Riefel induction technology).

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2.1 Prequantization

We assume that the reader is somewhat familiar with the ideas of geometric quantization [59, 1], so we will mainly establish our notation in this subsection. Interestingly, the argument runs slightly differently depending on which sign conventions one uses for Hamiltonian vector fields. We start by using the conventions mostly used by mathematicians (which, indeed, are the ones employed in [52, 53]). Here the Hamiltonian vector field ξ_f of $f \in C^\infty(S)$ is defined by $i_{\xi_f}\omega = -df$, where ω is the symplectic form on S. Similarly, the generator f_ξ of a vector field ξ, whose flow leaves ω invariant, is defined by $i_\xi\omega = -df_\xi$. The Poisson bracket is $\{f, g\} = \omega(\xi_f, \xi_g) = \xi_f g$. This implies that $[\xi_f, \xi_g] = \xi_{\{f,g\}}$ and $\{f_{\xi_1}, f_{\xi_2}\} = f_{[\xi_1,\xi_2]}$ (plus a possible central extension).

In geometric quantization one attempts to find a line bundle L over S with connection A and curvature F_A, satisfying

$$F_A = -\frac{i}{\hbar} pr^*\omega, \tag{2.1}$$

where $pr : \mathsf{L} \to S$ is the canonical projection. For $\hbar = 1/2\pi$ this is the condition $c_1(\mathsf{L}) = [\omega]$ stating that the Chern class of the line bundle equals the cohomology class of the symplectic form, cf. [16]. For any $f \in C^\infty(S)$, the prequantization $\pi_{\rm pre}(f)$ is an (unbounded) operator defined on the linear space of smooth sections of L with compact support; this space has a natural inner product derived from the Liouville measure on S (if $\dim S = n$ this measure corresponds to the volume form ω^n), and the completion may be identified with $L^2(S)$. The prequantization is defined by

$$\pi_{\rm pre}(f) = \frac{\hbar}{i}\nabla_{\xi_f} + f, \tag{2.2}$$

where ∇ is the covariant derivative defined by the connection A, and f is a multiplication operator. The crucial property satisfied by prequantization is

$$[\pi_{\rm pre}(f), \pi_{\rm pre}(g)] = -i\hbar\pi_{\rm pre}(\{f, g\}). \tag{2.3}$$

If a Lie group G acts on S, we may define a vector field ξ_X for each $X \in \mathbf{g}$ by $(\xi_X f)(s) = d/dt f(\exp(-tX)s)_{t=0}$. Writing f_X for f_{ξ_X}, one finds that $X \to f_X$

and $X \to (i/\hbar)\pi_{\rm pre}(f_X)$ are Lie algebra homomorphisms up to a possible central extension.

2.2 Prequantization of $\mathbb{C}P^n$

We will now prequantize the projective space of $\mathcal{H} = \mathbb{C}^{n+1}$. We choose $S = \mathbb{P}\mathcal{H} = \mathbb{C}P^n$. We define its symplectic structure through Marsden-Weinstein reduction, cf. the Introduction (for a direct definition cf. [16]). We start from $\mathcal{H}$, equipped with symplectic form $\tilde{\omega}(\psi,\varphi) = -2\hbar\,\mathrm{Im}\,(\psi,\varphi)$. The space $\mathbb{S}\mathcal{H} = \{\psi \in \mathcal{H} | (\psi,\psi) = 1\}$ is co-isotropically embedded in $\mathcal{H}$; the quotient by its null foliation is $\mathbb{P}\mathcal{H}$, which consists of equivalence classes $[\psi]$ in $\mathbb{S}\mathcal{H}$, where $\psi_1 \sim \psi_2$ iff $\psi_1 = z\psi_2$ for some $z \in \mathbb{C}$ with $|z| = 1$. The symplectic form ω on $\mathbb{P}\mathcal{H}$ is then the reduction of $\tilde{\omega}$, cf. [1, 17].

Let pr be the canonical projection from $\mathbb{S}\mathcal{H}$ to $\mathbb{P}\mathcal{H}$. This projection makes $\mathbb{S}\mathcal{H}$ a principal fibre bundle over $\mathbb{P}\mathcal{H}$ with structure group $U(1)$. We denote the generator of $U(1)$ by T ($T = i$ in the defining representation of $U(1)$ on $\mathbb{C}$). The vertical vector $v_T(\psi)$ at $\psi \in \mathbb{S}\mathcal{H}$ is $i\psi$, where we have identified $T_\psi\mathbb{S}\mathcal{H} \subset T_\psi\mathcal{H} \simeq \mathcal{H}$ with a subspace of $\mathcal{H}$ according to

$$T_\psi\mathbb{S}\mathcal{H} = \{\varphi \in \mathcal{H} | \mathrm{Re}\,(\psi,\varphi) = 0\}. \tag{2.4}$$

This bundle carries a connection A defined by

$$\langle A, \varphi\rangle(\psi) = \mathrm{Im}\,(\varphi,\psi) \otimes T. \tag{2.5}$$

Clearly, $\langle A, v_T\rangle(\psi) = T$ as required. It is then clear that the prequantization line bundle L is the hyperplane bundle H over $\mathbb{C}P^n$ [16]: this is the line bundle associated to the principal bundle $(\mathbb{S}\mathbb{C}^{n+1}, \mathbb{C}P^n, pr)$ by the representation $z \to \overline{z}$, or $T \to -i$, of $U(1)$. (For one may extend the tangent vectors $\varphi_1, \varphi_2 \in T_\psi\mathbb{S}\mathcal{H}$ to vector fields in a neighbourhood of ψ satisfying $[\varphi_1, \varphi_2] = 0$, and then the formula $\langle dA|\varphi_1, \varphi_2\rangle = \varphi_1\langle A, \varphi_2\rangle - \varphi_2\langle A, \varphi_1\rangle - \langle A, [\varphi_1, \varphi_2]\rangle$ and the replacement $T \to -i$ in the definition of A shows that (2.1) is satisfied.)

2.3 Quantization of $\mathbb{C}P^n$

To pass from prequantization to quantization we use the anti-holomorphic polarization on $\mathbb{C}P^n$; in local co-ordinates this is the distribution F on $\mathbb{C}P^n$ which is spanned by $\{\partial/\partial\overline{z_i}\}_i$. Since the connection is analytic [16], this polarization determines the polarized sections of H as the holomorphic ones. The space $\Gamma_{\text{hol}}(\mathsf{H})$ of holomorphic sections of H is well known (e.g., [16]): realizing the sections of H as equivariant functions

$$\Gamma_{\text{hol}}(\mathsf{H}) \ni \Psi : \mathbb{S}\mathbb{C}^{n+1} \to \mathbb{C}; \quad \Psi(\psi z) = z\Psi(\psi) \tag{2.6}$$

for all $z \in U(1)$, the holomorphic ones are in one-to-one correspondence with vectors $\varphi \in \mathbb{C}^{n+1}$, and given by $\Psi_\varphi(\psi) = (\psi, \varphi)$. Hence we obtain a linear map $V : \overline{\mathcal{H}} \to \Gamma_{\text{hol}}(\mathsf{H})$ from the conjugate space of $\mathcal{H}$ to the Hilbert space $\mathcal{H}_{\text{qua}} = \Gamma_{\text{hol}}(\mathsf{H})$ of the geometric quantization of $S = \mathbb{P}\mathcal{H}$, given by $(V\varphi)(\psi) = (\psi, \varphi)$. The inner product on $\Gamma_{\text{hol}}(\mathsf{H})$ is given by the Hermitian structure of the line budle H and integration over $\mathbb{P}\mathcal{H}$ w.r.t. the Liouville measure obtained from the symplectic form - there is no need for half-densities or so in this case. Normalizing the Liouville measure suitably, it follows that the map V is unitary (note that the inner product on $\overline{\mathcal{H}}$ is the complex conjugate of that on $\mathcal{H}$).

The final step in the geometric quantization of $\mathbb{P}\mathcal{H}$ (omitted in [52, 53]) is the quantization of (a subset of) the observables, i.e., the smooth functions on $S = \mathbb{P}\mathcal{H} = \mathbb{C}P^n$. Only those functions $f \in C^\infty(S)$ are quantizable which satsify the condition that $\pi_{\text{pre}}(f)\Psi \in \Gamma_{\text{hol}}(\mathsf{H})$ for all $\Psi \in \Gamma_{\text{hol}}(\mathsf{H})$. This is equivalent to the requirement that $[\xi_f, \xi] \in F$ for all $\xi \in F$. Hence ξ_f generates a holomorphic diffeomorphism of $\mathbb{C}P^n$ (the vector field is automatically complete because $\mathbb{C}P^n$ is compact).

In a move analogous to the proof of Wigner's theorem in [53], we now use Chow's theorem [16], which implies that any holomorphic diffeomorphism of $\mathbb{C}P^n$ is induced by an invertible linear transformation of $\mathbb{C}^n$. If we realize $\mathbb{C}P^n$ as $\mathbb{C}^{n+1}/\mathbb{C}^*$, and denote the corresponding projection from $\mathbb{C}^{n+1}$ to $\mathbb{C}P^n$ by pr, this corollary of Chow's theorem means that $\xi_f(pr(\psi)) = -pr_* X\psi$, where $X \in \mathbf{gl}_n(\mathbb{C})$ and $X\psi \in T_\psi\mathbb{C}^{n+1} \simeq \mathbb{C}^{n+1}$. But we know in addition that ξ_f is the Hamiltonian vector field of a (real-valued) function in $C^\infty(\mathbb{C}P^n)$; in particular, ξ_f must leave the symplectic form

invariant. Hence $X^* = -X$, and the flow of ξ_f is induced by unitary transformations $\exp(tX)$ of $\mathbb{C}^{n+1}$. Therefore, $X\psi$ is tangent to $S\mathbb{C}^{n+1}$, cf. (2.4), and we may return to our characterization of $\mathbb{C}P^n$ as $S\mathbb{C}^{n+1}/U(1)$. A simple exercise shows that the function producing this ξ_f as its Hamiltonian vector field is given by

$$f_X([\psi]) = i\hbar(X\psi, \psi), \tag{2.7}$$

where $\psi \in \mathbb{C}^{n+1}$ is an arbitrary preimage of $[\psi] \in \mathbb{C}P^n$. Conversely, the group $G = U(n+1)$ has a symplectic action on $\mathbb{C}P^n$ obtained by projecting its defining action on $\mathbb{C}^{n+1}$. For each $X \in \mathbf{u}_{n+1}$ the function f_X is then defined as explained after (2.3).

Before clarifying the significance of the result (2.7), we will describe the quantization $\pi_{\rm qua}(f_X)$; this is just the restriction of $\pi_{\rm pre}(f_X)$ to $\mathcal{H}_{\rm qua} = \Gamma_{\rm hol}(\mathsf{H})$. With $pr : \mathbb{S}\mathcal{H} \to \mathbb{P}\mathcal{H}$ we exploit the fact that $\xi_X(pr(\psi)) = -pr_* X\psi$, where $X\psi \in T_\psi \mathbb{S}\mathcal{H}$, cf. (2.4). With $\Psi \in \Gamma_{\rm hol}(\mathsf{H})$ realized as in (2.6), the covariant derivative acts according to $\nabla_\xi \Psi(\psi) = (\tilde{\xi} - v_{\langle A, \tilde{\xi}\rangle})\Psi(\psi)$, where $\tilde{\xi} \in T_\psi \mathbb{S}\mathcal{H}$ is an arbitrary lift of $\xi \in T_{pr(\psi)}\mathbb{P}\mathcal{H}$. With $\xi(\psi) = \xi_X(\psi)$ we of course choose the lift $\tilde{\xi_X}(\psi) = -X\psi$. Using (2.5) and (2.7) one finds that with this choice $\hbar\langle A, \tilde{\xi_X}\rangle = T \otimes f_X$. With (2.6) and the fact that $T = -i$ on the hyperplane bundle H, we find that the multiplication operator f in (2.2) cancels the term in ∇_{ξ_X} that comes from the connection A. At the end of the day we therefore obtain

$$(\pi_{\rm qua}(f_X)\Psi)(\psi) = i\hbar \frac{d}{dt}\Psi(e^{tX}\psi)_{t=0}. \tag{2.8}$$

As explained after (2.3), we can extract a representation of the Lie algebra of $U(\mathcal{H}) = U(n+1)$, which in this case exponentiates to a representation $\pi_{\rm qua}$ of $U(\mathcal{H})$. Realized on $\overline{\mathcal{H}} = V^{-1}\mathcal{H}_{\rm qua}$, we find from (2.8) that $\pi_{\rm qua}(U) = \overline{U}$.

We recall the steps leading to this result: the defining representation of $U(\mathcal{H})$ on $\mathcal{H}$ induces a symplectic action on $\mathbb{P}\mathcal{H}$, which is generated by the functions f_X. These can be quantized, which leads to a representation of $\mathbf{u}(\mathcal{H})$, which in turn is exponentiated to $\pi_{\rm qua}(U(\mathcal{H}))$. That the latter is the conjugate of the action on $\mathcal{H}$ we started from was to be expected from the identification of $\mathcal{H}_{\rm qua}$ with $\overline{\mathcal{H}}$. As we

shall see in subsection 2.5, this curious conjugation is merely a consequence of the sign conventions we have chosen (following [52, 53]).

2.4 More on the observables of quantum mechanics

The description of the observables of quantum mechanics as those (smooth) functions (2.7) on $\mathbb{P}\mathcal{H}$ that can be quantized in the anti-holomorphic polarization may not be their most compelling characterization. A physically more meaningful property of the function f_X (where $X \in \mathbf{u}_{n+1}$) is that it can be extended to an affine function on the state space K of the C^*-algebra $M_{n+1}(\mathbb{C})$ of linear operators on $\mathcal{H} = \mathbb{C}^{n+1}$. This state space consists of all normalized positive linear functionals on $M_{n+1}(\mathbb{C})$, hence each element ω of K satisfies $\omega(\mathbb{I}) = 1$ and $\omega(A^*A) \geq 0$. K is a compact convex set whose extreme boundary of pure states is the 'phase space' $\mathbb{C}P^n$. The embedding of $\mathbb{C}P^n = \mathbb{P}\mathcal{H}$ into K is obtained by realizing that a unit vector $\Omega \in \mathcal{H}$ defines a state ω by $\omega(A) = (A\Omega, \Omega)$. Each mixed state ω in K admits a (highly nonunique) extremal decomposition $\omega = \sum_i p_i \omega_i$ (with $\sum_i p_i = 1$) as a convex sum of pure states $\omega_i \in \mathbb{P}\mathcal{H}$.

A visually accessible example is provided by $\mathcal{H} = \mathbb{C}^2$, so that $\mathbb{P}\mathcal{H} = \mathbb{C}P^1 = S^2$. The state space of $M_2(\mathbb{C})$ (the algebra of 2×2 matrices) is the unit ball B^3 in $\mathbb{R}^3$; its extremal boundary, the two-sphere S^2 with unit radius, is the pure state space. Points in the interior may be writen as convex sums of boundary points in many ways.

A skew-adjoint operator X defines a continuous real-valued function f_X on K by $f_X(\omega) = i\omega(X)$; when restricted to the pure state space this function clearly coincides with (2.7). Conversely, $f_X \in C(K)$ is the unique affine extension of $f_X \in C^\infty(\mathbb{P}\mathcal{H})$ (a function f on K is called affine if $f(\lambda\omega_1 + (1-\lambda)\omega_2) = \lambda f(\omega_1) + (1-\lambda)f(\omega_2)$ for all $\omega_1, \omega_2 \in K$ and $0 < \lambda < 1$). An affine function on K is uniquely determined by its values on $\mathbb{P}\mathcal{H}$. However, a generic function on $\mathbb{P}\mathcal{H}$ cannot be extended to an affine function on K, because different extremal decompositions of a point in K would produce different values of the (extended) function at that point. The (relatively few) functions on $\mathbb{P}\mathcal{H}$ which are insensitive to this nonuniqueness are

precisely the 'linear' observables f_X of quantum mechanics. On $\mathbb{P}C^2 = \mathbb{C}P^1$ there are only four such (linearly independent) observables! (See [4] for the general theory of affine function spaces on compact convex sets.)

An alternative characterization of these observables f_X comes from the transformations $\varphi_t^X = \exp(t\xi_X)$ of $\mathbb{P}\mathcal{H}$ they generate via their Hamiltonian vector fields ξ_X. We have already seen that φ_t^X leaves the symplectic as well as the complex (and thereby the Kähler) structure of $\mathbb{P}\mathcal{H}$ invariant, cf. [11]. This implies that the transition probability (which on $\mathcal{H}$ is given by $|(\psi, \varphi)|^2$, and quotients to $\mathbb{P}\mathcal{H}$) is invariant under the flow φ_t^X of f_X. Conversely, Wigner's theorem implies that any transformation of $\mathbb{P}\mathcal{H}$ with this property is generated by a function of the type f_X (possibly composed with the anti-symplectic transformation on $\mathbb{P}\mathcal{H}$ which is induced from the map $\psi \to \overline{\psi}$ on $\mathcal{H}$), cf. [53]. A theorem of Shultz [49] then allows us to characterize the observables as those continuous functions on $\mathbb{P}\mathcal{H}$ whose flow is the restriction to the pure state space $\mathbb{P}\mathcal{H}$ of an affine homeomorphism of the total state space K. Finally, the equivalence of all descriptions listed is then confirmed by Kadison's theorem [23] that any affine homeomorphism φ_t of the state space K of a C^*-algebra $\mathfrak{A}$ is induced by a Jordan morphism of $\mathfrak{A}$; in the present case $\mathfrak{A} = M_{n+1}(\mathbb{C})$ this implies that φ_t must be induced by a unitary- or an anti-unitary operator on $\mathcal{H} = \mathbb{C}^{n+1}$.

Note, that the f_X form a subset of the Poisson algebra $C^\infty(\mathbb{P}\mathcal{H})$, but not a Poisson subalgebra: the relevant commutative multiplication is not the pointwise one used in classical mechanics, but the Jordan product $f_X \circ f_Y = \frac{1}{2} i\hbar f_{XY+YX}$. This product may be motivated by non-commutative spectral theory on convex sets [4], or by considerations involving the Kähler geometry of the pure state space [11].

With the exception of the compactness of K, all considerations in this subsection are equally well valid for $n = \infty$, if $M_\infty(\mathbb{C})$ is taken to be the C^*-algebra of compact operators. We see that from a physical point of view it is the affine structure of the total state space, rather than the complex structure of the pure state space, which is essential.

2.5 New sign conventions

We will actually recover $\mathcal{H}$ (rather than $\overline{\mathcal{H}}$) from the geometric quantization of $\mathbb{P}\mathcal{H}$ if we follow the conventions of [1], and define the Hamiltonian vector field ξ_f of $f \in C^\infty(S)$ by $i_{\xi_f}\omega = df$. The Poisson bracket is now $\{f, g\} = \omega(\xi_f, \xi_g) = -\xi_f g$. This implies that $[\xi_f, \xi_g] = -\xi_{\{f,g\}}$. If a Lie group G acts on S, we redefine the vector field ξ_X for each $X \in \mathbf{g}$ by $(\xi_X f)(s) = d/dt f(\exp(tX)s)_{t=0}$. Then $[\xi_X, \xi_Y] = -\xi_{[X,Y]}$, but, as with the old conventions, $\{f_X, f_Y\} = f_{[X,Y]}$ (plus a possible central extension). For geometric quantization these conventions imply that the connection on the prequantization bundle now has to satisfy $F_A = (i/\hbar)pr^*\omega$, rather than (2.1). The prequantization itself is still given by (2.2). Instead of (2.3), one now has $[\pi_{\mathrm{pre}}(f), \pi_{\mathrm{pre}}(g)] = i\hbar\pi_{\mathrm{pre}}(\{f, g\})$. Hence we obtain a (projective) representation of $\mathbf{g}$ by $X \to (-i/\hbar)\pi_{\mathrm{pre}}(f_X)$.

Since we have not changed the symplectic form ω on $\mathbb{P}\mathcal{H}$, the prequantization bundle is now obviously the tautological line bundle T over $\mathbb{P}\mathcal{H}$ [16], which is associated to the principal bundle $\mathbb{S}\mathcal{H}$ over $\mathbb{P}\mathcal{H}$ via the defining representation of $U(1)$ on $\mathbb{C}$. The space of holomorphic sections of T being empty [16], we now choose the holomorphic polarization on $\mathbb{P}\mathcal{H}$ to go from prequantization to quantization. The antiholomorphic sections of T are all of the form $\Psi(\psi) = (\varphi, \psi)$ for some $\varphi \in \mathcal{H}$, so that we find a unitary map $V : \mathcal{H} \to \mathcal{H}_{\mathrm{qua}}$ given by $(V\varphi)(\psi) = (\varphi, \psi)$. The quantization of f_X is given by *minus* (2.8). The representation of $U(\mathcal{H})$ defined through this quantization (cf. the text following (2.8)) is now simply the defining representation on $\mathcal{H}$. Hence geometric quantization has indeed recovered $\mathcal{H}$ from $\mathbb{P}\mathcal{H}$!

3 Representations of the unitary group from Rieffel induction

Our main purpose in this chapter is to obtain all irreducible unitary representations of the unitary groups $U(\mathcal{H})$ and $U_0(\mathcal{H})$ (cf. subsection 1.3) from an induction construction. In subsections 3.1 to 3.3 we take $\mathcal{H}$ to be an infinite-dimensional sep-

arable Hilbert space, unless explicitly stated otherwise. All results (sometimes with self-explanatory modifications) are equally well valid in the finite-dimensional case, which is considerably easier to handle; we leave this to the reader. We start with the simplest case, the defining representation, which forms the bridge between the preceding part of the paper and what follows.

3.1 The quantization of $\mathbb{P}\mathcal{H}$ revisited

As explained in the Introduction, we can realize $\mathbb{P}\mathcal{H}$ as a Marsden-Weinstein quotient. The group $U(1)$ acts on $\mathcal{H}$ (in principle from the right, though this is irrelevant here) by $\psi \rightarrow \psi z$, $\psi \in \mathcal{H}$, $|z| = 1$. The most general equivariant moment map $J_R : \mathcal{H} \rightarrow \mathbf{u}(\mathbf{1})^* = \mathbb{R}$ [1, 17] corresponding to this action is given by

$$J_R(\psi) = (\psi, \psi) + c, \tag{3.1}$$

where c is a constant (as explained in subsection 1.4, this is 'officially' an anti-Poisson homomorphism, but again this is irrelevant here). The reduced space $\mathcal{H}^1 = J^{-1}(1+c)/U(1)$ then coincides with with $\mathbb{P}\mathcal{H}$ as a symplectic space (that is, including the normalization of the symplectic form). We put $c = 0$ in what follows.

The quantization of this type of reduced space using Rieffel induction was outlined in subsection 1.4. We first need a quantization of the 'unconstrained' system $\mathcal{H}$, which we take to be the symmetric (bosonic) Fock space $\mathcal{F} = \exp(\mathcal{H})$ (this is the direct sum of all symmetrized tensor products $\mathcal{H}^{\otimes n}$ $(n = 0, 1, \ldots)$ of $\mathcal{H}$ with itself). This quantization is so well-established that we will not motivate it here; cf. [14, 45] for mathematical aspects, and [59] for a derivation in geometric quantization.

The (anti) representation π_R of $U(1)$ on $\mathcal{F}$ is obtained by 'quantization' of the right action on $\mathcal{H}$. No physicist would hesitate in choosing π_R as the second quantization of this right action. Labelling this choice $\pi_{R,\mathrm{sq}}$, this yields $\pi_{R,\mathrm{sq}}(z) \upharpoonright \mathcal{H}^{\otimes n} = z^n \mathbb{I}$. Similarly, the defining representation π_1 of $G = U(\mathcal{H})$ (the group of all unitary operators on $\mathcal{H}$) on $\mathcal{H}_1 = \mathcal{H}$ yields a symplectic action on $\mathcal{H}$. This is 'second' quantized by the representation $\pi_{L,\mathrm{sq}}$ on $\mathcal{F}$, whose restriction π_n to each subspace $\mathcal{H}^{\otimes n} \subset \mathcal{F}$ is the symmetrized n-fold tensor product of π_1 with itself. The representations $\pi_{R,\mathrm{sq}}(U(1))$ and $\pi_{L,\mathrm{sq}}(U(\mathcal{H}))$ obviously commute with each other. Hence

$\mathcal{F}$ has a central decomposition under $\pi_{L,\mathrm{sq}}(U(\mathcal{H})) \otimes \pi^{-1}_{R,\mathrm{sq}}(U(1))$, which is explicitly given by

$$\exp(\mathcal{H}) \stackrel{\mathrm{sq}}{\simeq} \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{U(\mathcal{H})} \otimes \overline{\mathcal{H}}_n^{U(1)}. \tag{3.2}$$

Here $\mathcal{H}_n^{U(\mathcal{H})}$ coincides with $\mathcal{H}^{\otimes n}$, now regarded as the carrier space of the representation $\pi_n(U(\mathcal{H}))$, which is, in fact, irreducible for all n [26, 40] (also cf. subsection 3.3 below). Also, $\mathcal{H}_n^{U(1)}$ is just $\mathbb{C}$, but regarded as the carrier space of $\pi_n(U(1))$, defined by $\pi_n(z) = z^n$; $\overline{\mathcal{H}}$ stands for the carrier space of the conjugate representation.

The general context for decompositions of the type (3.2) is the theory of Howe dual pairs [21, 19]. In the present instance, this applies to $\mathcal{H} = \mathbb{C}^k$, with $U(k)$ and $U(1)$ being the dual pair in $Sp(2k, \mathbb{R})$. (Cf. [43] for the theory of these pairs in the infinite-dimensional setting.)

The construction of the induced space $\mathcal{F}^1$ is effortless in this case. The fact that Marsden-Weinstein reduction took place at $J = 1$ means that the orbit of $U(1)$ in question is the point $1 \in \mathbf{u(1)}^*$. This orbit is quantized by the defining representation π_1 of $U(1)$ on $\mathcal{H}_1 = \mathbb{C}$. By construction, $\mathcal{F}^1$ is the subspace of $\mathcal{F} \otimes \mathcal{H}_1 = \mathcal{F}$ which is invariant under the representation $\pi_R^{-1} \otimes \pi_1$. Hence (3.2) implies that $\mathcal{F}^1 = \mathcal{H}$. The induced representation $\pi^1(U(\mathcal{H}))$ on $\mathcal{F}^1$ is simply the restriction of $\pi_{L,\mathrm{sq}}(U(\mathcal{H}))$ to this space, so that $\pi^1 \simeq \pi_1$. In other words, we have recovered the defining representation.

So far, so good, but unfortunately there is a subtlety if one derives π_R and π_L from geometric quantization. Using the 'uncorrected' formalism (as described, e.g., in Ch. 9 of [59]), exploiting the existence of an invariant positive totally complex polarization, viz. the anti-holomorphic one, one finds that $\mathcal{F}$ is realized as the space of holomorphic functions on $\mathcal{H}$. The quantization π_{qua} of the moment maps J_R for $U(1)$ and J_L for $U(\mathcal{H})$ (with respect to their respective actions on $\mathbb{C}^k$) then reproduce the second quantizations $\pi_{R,\mathrm{sq}}$ and $\pi_{L,\mathrm{sq}}$, respectively.

If, however, one is more sophisticated and incorporates the half-form correction to geometric quantization [59, Ch. 10], one obtains extra contributions: for $\mathcal{H} = \mathbb{C}^k$, $\pi_{\mathrm{qua}}(J_R)$ is replaced by $\pi_{\mathrm{qua}}(J_R) + k/2$, whereas $\pi_{\mathrm{qua}}(J_L)$ acquires an additional constant $\frac{1}{2}$ (times the unit matrix). These Lie algebra representations exponentiate

to unitary representations of double covers $\tilde{U}(k)$ and $\tilde{U}(1)$, which we denote by $\pi_{L,\mathrm{hf}}$ and $\pi_{R,\mathrm{hf}}$, respectively. Under $\pi_{L,\mathrm{hf}}(\tilde{U}(k)) \otimes \pi_{R,\mathrm{hf}}^{-1}(\tilde{U}(1))$ we then find the central decomposition

$$\exp(\mathcal{H}) \overset{\mathrm{hf}}{\simeq} \bigoplus_{n=0}^{\infty} \mathcal{H}^{\tilde{U}(k)}_{(n+\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})} \otimes \overline{\mathcal{H}}^{\tilde{U}(1)}_{n+\frac{1}{2}k}. \tag{3.3}$$

Here $\mathcal{H}_{(n+\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})}$ carries the representation of $\tilde{U}(k)$ with highest weight $(n+\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$; this is the tensor product of $\mathcal{H}_n$ and the square-root of the determinant representation. One observes that the inclusion of half-forms is awkward for Rieffel induction - we defer a discussion of this point to subsection 3.5

We finally turn to the question (discussed in subsection 2.3 in the context of geometric quantization) which observables are quantizable with the Rieffel induction method (in case it works, i.e., using $\pi_{L,\mathrm{sq}}$ and $\pi_{R,\mathrm{sq}}$!). For simplicity, in order to have bounded observables we restrict the algebra of classical observables $C^\infty(\mathcal{H})$ to $C_b^\infty(\mathcal{H})$, and take the quantization of the latter to be the self-adjoint part of the (von Neumann) algebra of all bounded operators $\mathfrak{B}(\mathcal{F})$ on $\mathcal{F}$. The subalgebra $\mathfrak{A}$ of operators whose Rieffel-induced representation π^1 on $\mathcal{F}^1$ can be defined is the commutant of $\pi_R(U(1))$ - this may be thought of as the quantization of the Poisson subalgebra A of $C_b^\infty(\mathcal{H})$ of functions invariant under $U(1)$, i.e., satisfying $f(\psi z) = \psi(z)$ for all $z \in U(1)$. (To explain a more intrinsic definition of $\mathfrak{A}$ we assume that the reader is familiar with the the theory of Hilbert C^*-modules and rigging maps, which play a role in the general theory of Rieffel induction [47, 56]. $\mathcal{F}$ is a Hilbert C^*-module for the group algebra $\mathfrak{B} = C^*(U(1))$. The rigging map $\langle\cdot,\cdot\rangle_{\mathfrak{B}}$ is given by [30] $\langle\Psi,\Phi\rangle_{\mathfrak{B}}(h) = (\pi_R^{-1}(h)\Phi,\Psi)$, which defines a continuous function on $U(1)$. The algebra $\mathfrak{A}$ is then the algebra of all adjointable maps on $\mathcal{F}$, in that each $A \in \mathfrak{A}$ satisfies $\langle A\Psi,\Phi\rangle_{\mathfrak{B}} = \langle\Psi,A^*\Phi\rangle_{\mathfrak{B}}$.)

In any case, the von Neumann algebra generated by $\pi_R(U(1))$ consists of the operators that are diagonal w.r.t. the decomposition $\mathcal{F} = \sum_n \mathcal{H}^{\otimes n}$. Hence by a well-known theorem of von Neumann stating that the commutant of the algebra of diagonalizable operators is the algebra of decomposable operators, $\mathfrak{A}$ consists of those bounded operators on $\mathcal{F}$ which map each $\mathcal{H}^{\otimes n}$ into itself. The induced representation $\pi^1(\mathfrak{A})$ is simply the restriction of $\mathfrak{A}$ to $\mathcal{H}$, so that $\pi^1(\mathfrak{A}) = \mathfrak{B}(\mathcal{H})$.

Thus we have not only produced the Hilbert space $\mathcal{H}$ as the quantization of $\mathbb{P}\mathcal{H}$, but in addition the correct algebra of observables has emerged.

3.2 The coadjoint orbits of $U_0(\mathcal{H})$ as reduced spaces

We recall that $G = U_0(\mathcal{H})$ is the Banach Lie group of all unitary operators U on $\mathcal{H}$ for which $U - \mathbb{I}$ is compact, equipped with the uniform operator (i.e., norm) topology. Its Lie algebra $\mathbf{g} = \mathbf{u}_0(\mathcal{H}) = i\mathfrak{K}(\mathcal{H})_{\rm sa}$ consists of all skew-adjoint compact operators on $\mathcal{H}$ with the norm topology. The dual $\mathbf{g}^* = \mathbf{u}_0(\mathcal{H})^*$ is the space of all self-adjoint trace-class operators on $\mathcal{H}$, with topology induced by the trace norm $\parallel \rho \parallel_1 = \mathrm{Tr}\,|\rho|$ (this coincides with the weak* topology). The pairing is given by $\langle \rho, X \rangle = i\,\mathrm{Tr}\,\rho X$.

The coadjoint action of $U_0(\mathcal{H})$ on $\mathbf{u}_0(\mathcal{H})^*$ is given by $\pi_{\rm co}(U)\rho = U\rho U^*$. We are interested in those coadjoint orbits which are 'quantizable' in the sense of geometric quantization, since their quantization should produce all irreducible representations of $U_0(\mathcal{H})$ [26, 27]. Each such orbit is labeled by a pair (m, n), where m is an ordered M-tuple of positive integers satisfying $m_1 \geq m_2 \geq \ldots m_M > 0$, and n is a similar N-tuple ($M, N < \infty$). The coadjoint orbit $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ consists of all elements of $\mathbf{u}_0(\mathcal{H})^*$ with eigenvalues $m_1, m_2, \ldots, m_M, 0^\infty, -n_N, \ldots, -n_1$. The degeneracy of each numerical eigenvalue m_i (or $-n_j$) is simply the number of times it occurs in this list. The explicit quantization of the orbits $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ is not discussed in [26, 27]; the case where either m or n is empty is done in [8] using geometric quantization.

For finite-dimensional $\mathcal{H}$, it was shown by Montgomery [38] that $\mathcal{O}_{\mathsf{m},0}$ can be written as a Marsden-Weinstein reduced space with respect to the natural right-action of $U(M)$ on $\mathcal{H} \otimes \mathbb{C}^M$. This is a special instance of the theory of dual pairs. With $\mathcal{H} = \mathbb{C}^k$, the groups $U(\mathcal{H})$ and $U(M)$ form a Howe dual pair inside the symplectic group $Sp(2kM, \mathbb{R})$ [21, 50, 19], and the moment maps J_R and J_L introduced below build a Weinstein dual pair, cf. [24, 57]. General theorems on the connection between coadjoint orbits of one group and Marsden-Weinstein reduced spaces w.r.t. the other group in a dual pair are given in [33]. We will now generalize the special case mentioned above to infinite-dimensional $\mathcal{H}$, and general orbits $\mathcal{O}_{\mathsf{m},\mathsf{n}}$.

We take $S = \mathcal{H} \otimes \mathbb{C}^{M+N}$, which we regard as a Hilbert manifold in the obvious way. We choose the canonical basis $\{e_i\}_{i=1,\ldots,M+N}$ in $\mathbb{C}^{M+N}$. The symplectic form ω on S is taken as (we put $\hbar = 1$)

$$\omega(\psi, \varphi) = -2\,\mathrm{Im}\left(\sum_{i=1}^{M}(\psi_i, \varphi_i) - \sum_{i=M+1}^{M+N}(\psi_i, \varphi_i)\right), \tag{3.4}$$

where we have expanded $\psi = \sum_i \psi_i \otimes e_i$ and similarly for φ. It is convenient to introduce an indefinite sesquilinear form on $\mathbb{C}^{M+N}$ by putting $(e_i, e_j) = \pm\delta_{ij}$, with a plus sign for $i = 1, \ldots, M$ and a minus sign for $i = M+1, \ldots, M+N$. Together with the inner product on $\mathcal{H}$ this induces an indefinite form $(\cdot, \cdot)_S$ on S in the obvious (tensor product) way. The r.h.s. of (3.4) then simply reads $-2\,\mathrm{Im}\,(\psi, \varphi)_S$. A simple trick shows that S is strongly symplectic: we can regard S as a Hilbert space $\mathcal{H} \otimes \mathbb{C}^M \oplus \overline{\mathcal{H} \otimes \mathbb{C}^N}$, with inner product $(\psi, \varphi)_{\text{trick}} = \sum_{i=1}^{M}(\psi_i, \varphi_i) + \sum_{i=M+1}^{M+N}(\varphi_i, \psi_i)$. Then $\omega(\psi, \varphi) = -2\,\mathrm{Im}\,(\psi, \varphi)_{\text{trick}}$, and the claim follows from the well-known fact that Hilbert spaces are strongly symplectic [34, 1].

The Lie group $H = U(M, N)$ (which is $U(M)$ or $U(N)$ for n or m empty) acts on S from the right in the obvious way, i.e., by $U \to \mathbb{I} \otimes U^T$. This action is symplectic, with anti-equivariant moment map $J_R : S \to (\mathbf{h}^*)^-$. If we identify $X \in \mathbf{h}$ with a generator in the defining representation of H on $\mathbb{C}^{M+N}$, we obtain (cf. [24, p. 501]

$$\langle J_R(\psi), X \rangle = i(\mathbb{I} \otimes X^T \psi, \psi)_S. \tag{3.5}$$

On a suitable Cartan subalgebra $\mathfrak{t}$ of $\mathbf{h}$, which we identify as the set of imaginary diagonal operators on $\mathbb{C}^{M+N}$, with basis $H_j = -iE_{jj}$, this simply reads $\langle J_R(\psi), H_j \rangle = \pm(\psi_j, \psi_j)$ with a plus sign for $j = 1, \ldots, M$ and a minus sign for $j = M+1, \ldots, M+N$.

We now identify (m, n) with an element of $\mathbf{h}^*$ by the pairing $\langle(\mathsf{m}, \mathsf{n}), X\rangle = i\mathrm{Tr}\, D_{(\mathsf{m},\mathsf{n})}X$, where $D_{(\mathsf{m},\mathsf{n})}$ is the diagonal matrix in $M_{M+N}(\mathbb{C})$ with entries $m_1, \ldots, m_M, -n_N, \ldots, -n_1$. This means that (m, n) defines a dominant integral weight on $\mathfrak{t}$, and vanishes on its complement. The subset $J_R^{-1}((\mathsf{m}, \mathsf{n}))$ of S is easily seen to consist of those $\psi = \sum_i \psi_i \otimes e_i$ for which

$$(\psi_1, \psi_1) = m_1, \ldots, (\psi_M, \psi_M) = m_M, (\psi_{M+1}, \psi_{M+1}) = n_N, \ldots, (\psi_{M+N}, \psi_{M+N}) = n_1,$$

and $(\psi_i, \psi_j) \sim \delta_{ij}$. The normalizations come from J_R evaluated on t, and the orhtogonality derives from the constraint that J_R vanish on its complement. *Note that the integrality of the m_i and n_j plays no role in this subsection.*

Lemma 1 $J_R^{-1}((\mathsf{m}, \mathsf{n}))$ *is a submanifold of* S.

Proof. According to the theorem on p. 550 of [10], we need to show that $J_R : J_R^{-1}((\mathsf{m}, \mathsf{n})) \to \mathbf{h}^*$ is a submersion, which is the case if at any point $\psi \in J_R^{-1}((\mathsf{m}, \mathsf{n})) \subset S$ the derivative $(J_R)_* \equiv J_R^{(1)} : T_\psi S \to T_{J_R(\psi)}\mathbf{h}^* \simeq \mathbf{h}^*$ is surjective and has a complementable kernel. The former is equivalent to the statement that ψ is a regular value of the moment map [1]. The derivative at $\psi \in S$ follows from (3.5) as

$$\langle (J_R^{(1)})_\psi(\xi), X\rangle = 2\mathrm{Re}\,(\mathbb{I} \otimes iX^T\xi, \psi)_S. \tag{3.6}$$

This formula shows that $J_R^{(1)}$ is continuous, so that its kernel is closed. The complementability of this kernel is then immediate, since S is a Hilbert manifold. The surjectivity of $J_R^{(1)}$ follows from (3.6) by inspection, but it is more instructive to derive it from Prop. 2.11 (due to Smale) in [35]. This states that ψ is a regular value of the moment map iff the stability group $H_\psi \subseteq H$ of ψ is discrete. Now, as pointed out earlier, $\psi = \sum_i \psi_i \otimes e_i \in J_R^{-1}((\mathsf{m}, \mathsf{n}))$ implies that all ψ_i are nonzero are orthogonal, so that H_ψ is just the identity. ■

The action of H on S is not proper unless m or n is empty (in which case H is compact). However:

Lemma 2 *The action of* H *on* $J_R^{-1}((\mathsf{m}, \mathsf{n}))$ *is proper.*

Proof. Let $\psi^{(n)} \to \psi$ in S; equivalently, $\psi_i^{(n)} \to \psi_i$ in $\mathcal{H}$ for all i. If $\{U^{(n)}\}$ is a sequence in H and $U^{(n)}\psi^{(n)}$ converges, the fact that for each n all $\psi_i^{(n)}$ are nonzero and orthogonal implies that $\{U_{ij}^{(n)} e_j\}$ must converge in $\mathbb{C}^{M+N}$ for each i. Since convergence in the topology on $U(M, N)$ is given by convergence of all matrix elements in the defining representation, this implies that $\{U^{(n)}\}$ must converge in H. ■

By the standard theory of Marsden-Weinstein reduction [34, 1], these lemmas imply that the reduced space

$$S^{(\mathsf{m},\mathsf{n})} = J_R^{-1}((\mathsf{m}, \mathsf{n}))/H_{(\mathsf{m},\mathsf{n})} \tag{3.7}$$

(where $H_{(\mathsf{m},\mathsf{n})}$ is the stability group of $(\mathsf{m},\mathsf{n}) \in \mathbf{h}^*$ under the coadjoint action) is a smooth symplectic manifold. We will proceed to show that it is symplectomorphic to the coadjoint orbit $\mathcal{O}_{\mathsf{m},\mathsf{n}} \in \mathbf{g}^*$, where $G = U_0(\mathcal{H})$, as explained above. The required diffeomorphism is given by a quotient of the moment map $J_L : S \to \mathbf{g}^*$ defined from the natural left-action of G on S, which action is evidently symplectic. Identifying $\mathbf{g}$ with the space of compact skew-adjoint operators Y on $\mathcal{H}$, one easily finds that this moment map is given by

$$-i\langle J_L(\psi), Y\rangle = (Y \otimes \mathbb{I}\psi, \psi)_S = \sum_{i=1}^{M}(Y\psi_i, \psi_i) - \sum_{i=M+1}^{M+N}(Y\psi_i, \psi_i). \qquad (3.8)$$

Since the left-G action and the right-H action commute, J_L is invariant under H (i.e., $J_L(\psi U) = J_L(\psi)$ for all $U \in H$ and $\psi \in \mathcal{H}$), so that J_L (restricted to $J_R^{-1}((\mathsf{m},\mathsf{n}))$) quotients to a well-defined map $\tilde{J}_L : S^{(\mathsf{m},\mathsf{n})} \to \mathcal{O}_{\mathsf{m},\mathsf{n}}$. Once we have shown that $\tilde{J}_L$ is a diffeomorphism, it will follow that it is symplectic, because of the definition of the symplectic structure on $S^{(\mathsf{m},\mathsf{n})}$ and the fact that J_L is equivariant.

Generalizing a standard result in the root and weight theory for compact Lie groups, see e.g. [5], we first note that the the stability group of $(\mathsf{m},\mathsf{n}) \in \mathbf{h}^*$ under the coadjoint action is $H_{(\mathsf{m},\mathsf{n})} = \prod_l U(l)$, where $\sum l = M + N$, and the product is over the multiplicities within either m or n in (m,n); this is a subgroup of $U(M,N)$ in the obvious block-diagonal form. (For example, if $(\mathsf{m},\mathsf{n}) = ((2,1,1),(2,2,2))$ the stability group is $U(1) \times U(2) \times U(3)$.) It then follows from (3.8) that $\tilde{J}_L$ is a bijection onto $\mathcal{O}_{\mathsf{m},\mathsf{n}}$.

Proposition 1 *$\tilde{J}_L$ is smooth.*

Proof. The manifold structure of $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ is defined by its embedding in $\mathbf{g}^*$, which is a Banach space in the trace-norm topology (cf. the beginning of this section). The smoothness of $\tilde{J}_L$ then follows from that of $J_L : J_R^{-1}((\mathsf{m},\mathsf{n})) \to \mathbf{g}^*$, since the Lie group H acts smoothly, freely, and properly on $J_R^{-1}((\mathsf{m},\mathsf{n}))$.

1. Continuity of J_L. We prove continuity on all of S. As a map between separable metric spaces (S is separable because $\mathcal{H}$ is by assumption, and $\mathbf{g}^*$ is separable because the finite-rank operators are dense in it), J_L is continuous if

$\psi^{(n)} \to \psi$ in S implies $J_L(\psi^{(n)}) \to J_L(\psi)$ in $\mathbf{g}^*$. The topology on $\mathbf{g}^*$ coincides with the weak*-topology, so the desired continuity follows from (3.8), the boundedness of Y, and Cauchy-Schwartz.

2. Existence and continuity of $J_L^{(1)}$. The derivative of J_L at ψ is given by

$$\langle (J_L^{(1)})_\psi(\xi), Y\rangle = 2\mathrm{Re}\left(\sum_{i=1}^{M}(iY\xi_i,\psi_i) - \sum_{i=M+1}^{M+N}(iY\xi_i,\psi_i)\right). \tag{3.9}$$

By the same reasoning as in the previous item, $(J_L^{(1)})_\psi$ lies in $\mathcal{L}(S,\mathbf{g}^*)$ and is continuous.

The second derivative $J_L^{(2)} : S \times S \to \mathbf{g}^*$ can be read off from (3.9); its existence and continuity are established as before. Higher derivatives vanish. ■

Proposition 2 *$\tilde{J}_L^{-1}$ is smooth.*

Proof. We pick an arbitrary point $\rho_0 \in \mathcal{O}_{\mathsf{m},\mathsf{n}}$, with stability group G_0. Let $\mathcal{H} = \oplus_l \mathcal{H}_l$ be the decomposition of $\mathcal{H}$ under which ρ_0 is diagonal (the dimension of each $\mathcal{H}_0$ is the degeneracy of the corresponding eigenvalue; this dimension is finite unless the eigenvalue is 0). Then $G_0 = \oplus_l U_0(\mathcal{H}_l)$, in self-evident notation. The Lie algebra $\mathbf{g}_0$ of G_0 is given by those operators in $\mathbf{g} = i\mathfrak{K}(\mathcal{H})_{\mathrm{sa}}$ which commute with ρ_0. The manifold $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ is modelled on $\mathbf{g}/\mathbf{g}_0$. This has the quotient topology inherited from $\mathbf{g}$, i.e., the trace-norm topology determined by $\| A \|_1 = \mathrm{Tr}\,|A|$.

We define a neighbourhood $V_0 \subset \mathcal{O}_{\mathsf{m},\mathsf{n}}$ of ρ_0 as follows. Since G is a Banach-Lie group, by [32] there exists a neighbourhoud V of $0 \in \mathbf{g}$ such that exp is a diffeomorphism on V into $\mathbf{g}$. We put $V_0 = \{\pi_{\mathrm{co}}(\exp(A))\rho_0 | A \in V\}$ (recall that the coadjoint action is given by $\pi_{\mathrm{co}}(U)\rho = U\rho U^*$). To define a chart on V_0, we first show that $\mathbf{g}$ (equipped with the trace-norm topology) admits a splitting $\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{m}_0$. Here $\mathbf{m}_0$ consists of those operators A in $\mathbf{g}$ whose matrix elements $(A\psi,\varphi)$ vanish if both ψ and φ lie in the same space $\mathcal{H}_l$, for all l. It is clear that $\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{m}_0$ as a set, and it quickly folows that each summand is closed: since $\| A \| \leq \| A \|_1$, the uniform topology is weaker than the trace-norm one, so that closedness in the former implies the corresponding property in the latter topology. As to the uniform closedness of g_0, one has $\| [A,\rho_0] \| \leq 2 \| A \| \| \rho_0 \|$, so that $\mathbf{g}_0 \ni A_n \to A$ implies that $A \in \mathbf{g}_0$.

On $\mathbf{m}_0$ an even more elementary inequality does the job. Thus $\mathbf{g}/\mathbf{g}_0 \simeq \mathbf{m}_0$, and we may use $\mathbf{m}_0$ as a modelling space for $\mathcal{O}_{\mathsf{m},\mathsf{n}}$.

We define a chart on V_0 by $\varphi_0 : V_0 \to \mathbf{m}_0$, given by $\varphi_0(\pi_{\mathsf{co}}(\exp(A))\rho_0) = A_0$, where A_0 is the component of $A \in \mathbf{g}$ in $\mathbf{m}_0$. We would like to model $S^{(\mathsf{m},\mathsf{n})}$ on $\mathbf{m}_0$ as well, but this is not directly possible because it has the wrong topology. Hence the following detour. Take a $\psi_0 \in J_R^{-1}((\mathsf{m},\mathsf{n})) \subset S$ for which $J_L(\psi_0) = \rho_0$. Using the fact that J_L is a bijection, we model $S^{(\mathsf{m},\mathsf{n})} = J_R^{-1}((\mathsf{m},\mathsf{n}))/H_{(\mathsf{m},\mathsf{n})}$ on the closed linear subspace of S given by $M_0 = \{A \otimes \mathbb{I}\psi_0 | A \in \mathbf{m}_0\}$, equipped with the relative topology of S. Put $W_0 = \{\exp(A) \otimes \mathbb{I}\psi_0 | A \in m_0\} \subset S$. If $pr : J_R^{-1} \to J_R^{-1}((\mathsf{m},\mathsf{n}))/H_{(\mathsf{m},\mathsf{n})}$ is the canonical projection, we have a chart on the neighbourhood $pr(W_0)$ of $pr(\psi_0)$ defined by $\phi_0 : pr(W_0) \to M_0$ given by $\phi_0(pr(\exp(A)\psi_0)) = A\psi_0$. This procedure respects the manifold structure of $S^{(\mathsf{m},\mathsf{n})}$, which by definition is quotiented from $J_R^{-1}((\mathsf{m},\mathsf{n})) \subset S$.

We now define ${}_0\tilde{J}_L^{-1} = \phi_0 \circ \tilde{J}_L^{-1} \circ \varphi_0^{-1}$; this is a map from $\varphi_0(V_0) \subset \mathbf{m}_0$ to $\phi_0 \circ pr(W_0) \subset M_0$. Clearly, ${}_0\tilde{J}_L^{-1}(A) = A\psi_0$. This immediately implies that ${}_0\tilde{J}_L^{-1}$, and therefore $\tilde{J}_L^{-1}$, is smooth. ■

It would have been possible to prove Proposition 1 using the method of proof of Proposition 2, but that would necessitate an argument (more complicated than our direct proof of Proposition 1) to the effect that the trace-norm topology restricted to $\mathbf{m}_0$ is equivalent to the strong operator topology [7]. In contrast, in Proposition 2 we merely needed the continuity of the identity map on $\mathbf{m}_0$, with the trace-norm topology as the initial one, and the strong operator topology as the final one. This is trivial, for the trace-norm topology is finer than the uniform topology, which in turn is finer that the strong operator topology. To sum up, we have proved

Theorem 1 *For any separable Hilbert space $\mathcal{H}$, the coadjoint orbit $\mathcal{O}_{\mathsf{m},\mathsf{n}}$ of the group $U_0(\mathcal{H})$ (which consists of all trace-class operators on $\mathcal{H}$ with M specific positive and N specific negative eigenvalues) is symplectomorphic to the Marsden-Weinstein quotient $S^{(\mathsf{m},\mathsf{n})} = J_R^{-1}((\mathsf{m},\mathsf{n}))/H_{(\mathsf{m},\mathsf{n})}$ with respect to $S = \mathcal{H} \otimes \mathbb{C}^{M+N}$ and the natural right-action of $H = U(M,N)$.*

3.3 Representations induced from $U(M)$

The representations of $U_0(\mathcal{H})$ were fully classified in [26, 40, 42] (also cf. [27, 43, 9]). A remarkable fact is that $U_0(\mathcal{H})$ is a type I group, so that all its factorial representations are of the form $\pi \otimes \mathbb{I}$ on $\mathcal{H}_\pi \otimes \mathcal{H}_{\rm mult}$, where $(\pi, \mathcal{H}_\pi)$ is irreducible. Each irreducible representation corresponds to an integral weight (m, n) of the type specified above, where M and N are arbitrary (but finite). The carrier space $\mathcal{H}^{(\mathsf{m},\mathsf{n})}$ is of the form $\mathcal{H}^{(\mathsf{m},\mathsf{n})} = \mathcal{H}^{\mathsf{m}} \otimes \overline{\mathcal{H}}^{\mathsf{n}}$, and carries the irreducible representation $\pi^{(\mathsf{m},\mathsf{n})} = \pi^{\mathsf{m}} \otimes \overline{\pi}^{\mathsf{n}}$. Here $\mathcal{H}^{\mathsf{m}}$ is the subspace of $\otimes^M \mathcal{H}$ obtained by symmetrization according to the Young diagram whose k-th row has length m_k, and $\overline{\mathcal{H}}^{\mathsf{n}}$ is the conjugate space of $\mathcal{H}^{\mathsf{n}}$. The representation π^{m} is the one given by the restriction of the M-fold tensor product of the defining representation of $U_0(\mathcal{H})$ to $\mathcal{H}^{\mathsf{m}}$, etc.

This is almost identical to the theory for finite-dimensional $\mathcal{H} = \mathbb{C}^k$ [58, 61] (which has the obvious restriction that $M, N \leq k$); the only difference is that in the infinite-dimensional case $\mathcal{H}^{\mathsf{m}} \otimes \overline{\mathcal{H}}^{\mathsf{n}}$ is already irreducible. For $k < \infty$, on the other hand, one needs to take the so-called Young product [61] of $\mathcal{H}^{\mathsf{m}}$ and $\overline{\mathcal{H}}^{\mathsf{n}}$ rather than the tensor product (this is the irreducible subspace generated by the tensor product of the highest-weight vectors in each factor); moreover, the use of conjugate spaces may be avoided in that case by tensoring with powers of the determinant representation. For example, $\mathbb{C}^k \otimes \overline{\mathbb{C}}^k$ contains the irreducible subspace $\sum_{i=1}^k e_i \otimes \overline{e_i}$ which does not lie in the Young product; for $k = \infty$ this subspace evidently no longer exists. For $M = 0$ or $N = 0$ there is no difference whatsoever.

We will now show how the representations $(\pi^{\mathsf{m}}, \mathcal{H}^{\mathsf{m}})$ can be obtained by Rieffel induction; the representations $(\overline{\pi}^{\mathsf{n}}, \overline{\mathcal{H}}^{\mathsf{n}})$ may then be constructed similarly. This will quantize the coadjoint orbits $\mathcal{O}_{\mathsf{m}} \equiv \mathcal{O}_{(\mathsf{m},\emptyset)}$ and $\mathcal{O}^-_{\mathsf{n}} \equiv \mathcal{O}_{(\emptyset,\mathsf{n})}$, respectively. We note that $\mathcal{O}^-_{\mathsf{n}}$ is $\mathcal{O}_{\mathsf{n}}$ with the sign of the symplectic form changed; this relative minus sign corresponds to the passage from $\mathcal{H}$ to $\overline{\mathcal{H}}$ upon quantization.

Our starting point is Theorem 1, in which we take $S = \mathcal{H} \otimes \mathbb{C}^M$, with $H = U(M)$ acting on S from the right and $G = U_0(\mathcal{H})$ acting from the left in the natural way; we call these actions $\pi_1^T(H)$ and $\pi_1(G)$, respectively. As explained in part 1.4 of the Introduction, we first have to quantize S and the group actions defined on it. We do

so by taking the bosonic second quantization, or symmetric Fock space, $\mathcal{F} = \exp(S)$ over S [45, 59], cf. subsection 3.1. For later use, we equivalently define this as the subspace of $\sum_{n=0}^{\infty} \otimes^n S$ on which the natural representation of the symmetric group S_n on $\otimes^n S$ acts trivially for all n.

As in the $M = 1$ case (cf. subsection 3.1) we first investigate the representations of $U_0(\mathcal{H})$ and $U(k)$ on $\mathcal{F}$ obtained by second quantization, or equivalenty, by geometric quantization without the half-form modification. This goes as follows. The groups H and G act on each subspace $\otimes^n S$ by the n-fold tensor product of their respective actions on S, and these actions restrict to $\mathcal{F}$. Thus the actions $\pi_1^T(H)$ (which we turn into a representation by taking the inverse) and $\pi_1(G)$ on S are quantized by the unitary representations $\Gamma\overline{\pi}_1(H)$ $(= \pi_{R,\text{sq}}^{-1}(H)$ in the notation of subsection 3.1, and $\pi_R^{-1}(H)$ in that of subsection 1.4) and $\Gamma\pi_1(G)$ $(= \pi_{L,\text{sq}}(G))$, respectively (note that $\pi_1^T(h^{-1}) = \overline{\pi}_1(h)$). Here Γ is the second quantization functor [45]. This setup, and the associated central decomposition of $\mathcal{F}$ under these group actions, illustrate Howe's theory of dual pairs [21, 18, 19] in an infinite-dimensional setting, cf. [43].

The fact that the coadjoint orbit $\mathcal{O}_{\mathsf{m}}$ of G is (symplectomorphic to) the Marsden-Weinstein quotient of S with respect to $\mathsf{m} \in \mathbf{h}^*$, cf. Theorem 1, should now be reflected, or rather quantized, by constructing the unitary representation $\pi^{\mathsf{m}}(G)$ (which according to Kirillov is attached to $\mathcal{O}_{\mathsf{m}}$) by Rieffel induction from the representation $\pi_{\mathsf{m}}(H)$ attached to the orbit through m in H. Here $\pi_{\mathsf{m}}(U(M))$ is simply the unitary irreducible representation given by the highest weight m; it is realized on $\mathcal{H}_{\mathsf{m}}$, which is the subspace of $\otimes^M \mathbb{C}^M$ obtained by symmetrization according to the Young diagram whose k-th row has length m_k.

As mentioned in subsection 1.4, to find the carrier space of the induced representation $\pi^{\mathsf{m}}(G)$ we merely have to identify the subspace of $\mathcal{F} \otimes \mathcal{H}_{\mathsf{m}}$ which is invariant under $\Gamma\overline{\pi}_1 \otimes \pi_{\mathsf{m}}(H)$. This is very easy on the basis of the following well-known facts [58, 61, 20]:

1. The representations of the symmetric group S_n are self-conjugate; for any irreducible representation $\pi_{\mathsf{l}}(S_n)$, the tensor product $\pi_{\mathsf{l}} \otimes \pi_{\mathsf{l}}$ contains the identity

representation once, and $\pi_{\mathsf{l}} \otimes \pi_{\mathsf{l}'}$ does not contain the identity unless $\mathsf{l} = \mathsf{l}'$. (Recall that the irreducible representations of S_n are labelled by an n-tuple of integers $\mathsf{l} = (l_1, \ldots, l_n)$, where $l_1 \geq l_2 \geq \ldots l_n \geq 0$ and $\sum_i l_i = n$.) The collection of all such n-tuples l forms the dual $\hat{S}_n$.

2. Any unitary irreducible representation $\pi_{\mathsf{l}}(U(M))$ is given by an M-tuple $\mathsf{l} = (l_1, \ldots, l_M)$ of positive nondecreasing integers (possibly zero), as in the preceding item, or by the conjugate $\overline{\pi_{\mathsf{l}}}$ of such a representation. Then $\pi_{\mathsf{l}} \otimes \overline{\pi_{\mathsf{l}}}$ contains the identity representation once, but the identity does not occur in any $\pi_{\mathsf{l}} \otimes \pi_{\mathsf{l}'}$, or in any $\pi_{\mathsf{l}} \otimes \overline{\pi_{\mathsf{l}'}}$ unless in the latter case $\mathsf{l} = \mathsf{l}'$.

3. The defining representation of S_n on $\otimes^n \mathbb{C}^M$ commutes with the n-fold tensor product of the conjugate of the defining representation of $U(M)$, so that one has the central decomposition
$$\otimes^n \mathbb{C}^M \simeq \bigoplus_{\mathsf{l}' \in \hat{S}_n} \mathcal{H}_{\mathsf{l}}^{S_n} \otimes \overline{\mathcal{H}}_{\mathsf{l}}^{U(M)}, \tag{3.10}$$
where the prime (relevant only when $M < n$) on the $\oplus$ indicates that the sum is only over those n-tuples l for which $l_{M+1} = 0$. Here $\mathcal{H}_{\mathsf{l}}^{S_n}$ is the carrier space of $\pi_{\mathsf{l}}(S^n)$, and $\overline{\mathcal{H}}_{\mathsf{l}}^{U(M)}$ is the carrier space of the conjugate of the irreducible representation of $U(M)$ obtained by making l an M-tuple by adding or removing zeros. (A simliar statement holds without the conjugation, of course.)

4. Similarly,
$$\otimes^n \mathcal{H} \simeq \bigoplus_{\mathsf{l} \in \hat{S}_n} \mathcal{H}_{\mathsf{l}}^{S_n} \otimes \mathcal{H}^{\mathsf{m}}, \tag{3.11}$$
under the appropriate representations of S_n and $U_0(\mathcal{H})$, where $\mathcal{H}^{\mathsf{m}}$ was introduced at the beginning of this subsection (for $\mathcal{H} = \mathbb{C}^k$ this is equivalent to a classical result in invariant theory, see e.g. [20, 4.3.3.9]).

Now consider $\otimes^n(\mathcal{H} \otimes \mathbb{C}^M) \simeq \otimes^n \mathcal{H} \otimes \otimes^n \mathbb{C}^M$. This carries the representation $\pi_n^{\mathcal{H}} \otimes \pi_n^{\mathbb{C}^M}$ of S_n, where $\pi_n^{\mathcal{K}}(S_n)$ is the natural representation on $\otimes^n \mathcal{K}$. Applying

items 4 and 3, and subsequently 1 above, we find that the subspace $\otimes_s^n(\mathcal{H}\otimes\mathbb{C}^M)\subset\otimes^n(\mathcal{H}\otimes\mathbb{C}^M)$ which is invariant under S_n can be decomposed as

$$\bigotimes_s^n(\mathcal{H}\otimes\mathbb{C}^M)\simeq\bigoplus_{\mathsf{l}'\in\hat{S}_n}\mathcal{H}^{\mathsf{l}}\otimes\overline{\mathcal{H}}_{\mathsf{l}}^{U(M)}, \tag{3.12}$$

in the sense that the restriction $\otimes_s^n(\pi_1(G)\otimes\overline{\pi}_1(H))$ of $\Gamma\pi_1(G)\otimes\Gamma\overline{\pi}_1(H)$ (defined on $\mathcal{F}=\exp(\mathcal{H}\otimes\mathbb{C}^M)$) to $\otimes_s^n(\mathcal{H}\otimes\mathbb{C}^M)\subset\mathcal{F}$ decomposes as

$$\bigotimes_s^n(\pi_1(G)\otimes\overline{\pi}_1(H))\simeq\bigoplus_{\mathsf{l}'\in\hat{S}_n}\pi^{\mathsf{l}}(G)\otimes\overline{\pi_{\mathsf{l}}}(H). \tag{3.13}$$

We then apply item 2 to conclude that the only subspace of $\mathcal{F}\otimes\mathcal{H}_{\mathsf{m}}$ which is invariant under $\Gamma\overline{\pi}_1\otimes\pi_{\mathsf{m}}(H)$ corresponds to $n=\sum_{i=1}^M m_i$ (where m_i are the entries of the M-tuple m). Moreover, by (3.12) this invariant subspace is exactly $\mathcal{H}^{\mathsf{m}}$ as a $U_0(\mathcal{H})$ module. Hence we have proved

Theorem 2 *Regard the symmetric Fock space $\mathcal{F}=\exp(\mathcal{H}\otimes\mathbb{C}^M)$ as a left-module (representation space) of $U_0(\mathcal{H})$ and a right-module of $U(M)$ under the second quantization of their respective natural actions on $\mathcal{H}\otimes\mathbb{C}^M$. Applying Rieffel induction to this bimodule, inducing from the irreducible representation $\pi_{\mathsf{m}}(U(M))$ (which corresponds to the highest weight $\mathsf{m}=(m_1,\ldots,m_M)$), yields the induced space $\mathcal{H}^{\mathsf{m}}$ carrying the irreducible representation $\pi^{\mathsf{m}}(U_0(\mathcal{H}))$.*

This, then, is the exact quantum counterpart of Theorem 1, specialized to $\mathsf{n}=\emptyset$. As remarked earlier, there exists an obvious analogue of Theorem 2 for $\mathsf{m}=\emptyset$, in which all Hilbert spaces and representations occurring in the construction are replaced by their conjugates.

To prepare for the next subsection we will now give a slight reformulation of the proof. We start with finite-dimensional $\mathcal{H}=\mathbb{C}^k$, with $k>M$. Classical invariant theory [20] then provides the decomposition of $\exp(S)$ under $\Gamma\pi_1(U(k))\otimes\Gamma\overline{\pi}_1(U(M))$ as

$$\exp(\mathbb{C}^k\otimes\mathbb{C}^M)\overset{\text{sq}}{\simeq}\bigoplus_{\mathsf{l}\in D_M}\mathcal{H}_{\mathsf{l}}^{U(k)}\otimes\overline{\mathcal{H}}_{\mathsf{l}}^{U(M)}, \tag{3.14}$$

where the sum is over all Young diagrams (or tuples) D_M with M rows or less, including the empty diagram. (Note that it would have been consistent with our

previous notation to write $(\mathcal{H}^{\mathsf{l}})^{U(k)}$ for $\mathcal{H}_{\mathsf{l}}^{U(k)}$; both stand for the irreducible representation of $U(k)$ defined by the Young diagram l. In what follows, we will reserve the notation $\mathcal{H}^{\mathsf{l}}$ for $\mathcal{H}_{\mathsf{l}}(U_0(\mathcal{H}))$, where $\mathcal{H} = l^2$.) Eq. (3.14) is an illustration of the theory of Howe dual pairs [21, 18, 19]: it exhibits a multiplicity-free central decomposition of $\mathcal{F} = \exp(S)$ under the commuting actions of $U(k)$ and $U(M)$ (which form a dual pair in $Sp(2kM, \mathbb{R})$, of which $\mathcal{F}$ carries the metaplectic representation).

In order to study the limit $k \to \infty$ we realize $\exp(\mathcal{H} \otimes \mathbb{C}^M)$ (with $\mathcal{H} = l^2$ now infinite-dimesional) as an (incomplete) infinite tensor product [55] with respect to the vacuum vector $\Omega \in \exp(\mathbb{C}^M)$, that is (recalling $\exp(\mathbb{C}^k \otimes \mathbb{C}^M) \simeq \otimes^k \exp(\mathbb{C}^M)$), $\exp(\mathcal{H} \otimes \mathbb{C}^M) \simeq \otimes_{\Omega}^{\infty} \exp(\mathbb{C}^M)$, where the right-hand side is the Hilbert space closure (with respect to the natural inner product on tensor products) of the linear span of all vectors of the type $\psi_1 \otimes \dots \psi_l \otimes \Omega \otimes \Omega \dots$, $\psi_i \in \exp(\mathbb{C}^M)$, in which only finitely many entries differ from Ω. (The term 'incomplete' refers to the fact that only 'tails' close to an infinite product of Ω's appear.) Thus $\exp(\mathbb{C}^k \otimes \mathbb{C}^M) \simeq \otimes^k \exp(\mathbb{C}^M)$ is naturally embedded in $\exp(\mathcal{H} \otimes \mathbb{C}^M)$ by simply adding an infinite tail of Ω's, and this provides an embedding $\exp(\mathbb{C}^k \otimes \mathbb{C}^M) \subset \exp(\mathbb{C}^{k+1} \otimes \mathbb{C}^M)$ as well. Clearly, $\exp(\mathcal{H} \otimes \mathbb{C}^M)$ coincides with the closure of the inductive limit $\cup_{k=1}^{\infty} \exp(\mathbb{C}^k \otimes \mathbb{C}^M)$ defined by this embedding.

Choosing the natural basis in $\mathcal{H} = l^2$, we obtain an embedding $U(k) \subset U(k+1)$, with corresponding actions on $\mathcal{H}$; our group $U_0(\mathcal{H})$ (realized in its defining representation on $\mathcal{H}$) is the norm-closure of the inductive limit group $\cup_{k=1}^{\infty} U(k)$. Using the explicit realization of $\mathcal{H}^{\mathsf{l}}$ as a Young-symmetrized tensor product, we similarly obtain embeddings $\mathcal{H}_{\mathsf{l}}(U(k)) \subset \mathcal{H}_{\mathsf{l}}(U(k+1))$. Thus the inductive limit $\cup_{k=1}^{\infty} \mathcal{H}_{\mathsf{l}}(U(k))$ is well-defined. Using (3.14), we then have that $\exp(\mathcal{H} \otimes \mathbb{C}^M)$ is the closure of $\cup_{k=1}^{\infty} \oplus_{\mathsf{l} \in D_M} \mathcal{H}_{\mathsf{l}}^{U(k)} \otimes \overline{\mathcal{H}}_{\mathsf{l}}^{U(M)}$, which in turn coincides with the closure of $\oplus_{\mathsf{l} \in D_M} \cup_{k=1}^{\infty} \mathcal{H}_{\mathsf{l}}^{U(k)} \otimes \overline{\mathcal{H}}_{\mathsf{l}}^{U(M)}$. We now use the fact that the closure of $\cup_{k=1}^{\infty} \mathcal{H}_{\mathsf{l}}^{U(k)}$ is $\mathcal{H}^{\mathsf{l}}$ as a representation space of $U_0(\mathcal{H})$ (this is obvious given the explicit realization of these spaces, but it is a deep result that an analogous fact holds for all representations of

$U_0(\mathcal{H})$ [40, 42, 43]). This yields the desired decomposition

$$\exp(\mathcal{H}\otimes\mathbb{C}^M) \overset{\text{sq}}{\simeq} \bigoplus_{\mathsf{l}\in D_M} \mathcal{H}_{\mathsf{l}} \otimes \overline{\mathcal{H}}_{\mathsf{l}}^{U(M)}, \tag{3.15}$$

under $\Gamma\pi_1(U_0(\mathcal{H}))\otimes\Gamma\overline{\pi}_1(U(M))$. This result was previously derived in [43] using a technique of holomorphic extension of representations.

Starting from (3.15), Theorem 2 follows immediately from item 2 on the list of ingredients of our previous proof.

To end this subsection we register how the half-form correction to geometric quantization modifies (3.14), cf. subsection 3.1, and in particular (3.3). These corrections are finite only for $\mathcal{H}=\mathbb{C}^k$, $k<\infty$, so we only discuss that case. As for $M=1$, one finds that the half-form quantizations of the moment maps corresponding to the $U(k)$ and $U(M)$ actions on $\mathbb{C}^k\otimes\mathbb{C}^M$ lead to Lie algebra representations that can only be exponentiated to representations $\pi_{L,\text{hf}}$ and $\pi_{R,\text{hf}}^{-1}$ of the covering groups $\tilde{U}(k)$ and $\tilde{U}(M)$ of $U(k)$ and $U(M)$, respectively, on which the square-root of the determinant is defined. A straightforward exercise leads to the decomposition

$$\exp(\mathbb{C}^k\otimes\mathbb{C}^M) \overset{\text{hf}}{\simeq} \bigoplus_{\mathsf{l}\in D_M} \mathcal{H}_{\mathsf{l}+\frac{1}{2}M}^{\tilde{U}(k)} \otimes \overline{\mathcal{H}}_{\mathsf{l}+\frac{1}{2}k}^{\tilde{U}(M)} \tag{3.16}$$

under $\pi_{L,\text{hf}}(\tilde{U}(k))\otimes\pi_{R,\text{hf}}^{-1}(\tilde{U}(M))$. Here $\mathsf{l}+\frac{1}{2}M$, regarded as a highest weight, has components $(l_1+\frac{1}{2}M, l_2+\frac{1}{2}M,\ldots)$, and analogously for $\mathsf{l}+\frac{1}{2}k$. Hence $\mathcal{H}_{\mathsf{l}+\frac{1}{2}M}$ carries the tensor product of the representation of $\tilde{U}(k)$ characterized by the Young diagram l, and the determinant representation to the power $M/2$, etc. This will be further discussed in subsection 3.5.

3.4 Representations induced from $U(M,N)$

We are now going to attempt to 'quantize' Theorem 1 for $N\neq 0$. The first problem is finding a unitary representation of $H=U(M,N)$ that corresponds to the dominant integral weight (m,n) on t (or the corresponding coadjoint orbit in $\mathbf{h}^*$, cf. subsection 3.2); this is the representation we should induce from. This problem was solved in [2], partly on the basis of the classification of all unitary highest-weight modules of $U(M,N)$ [12, 22, 41]. In the compact case, each dominant integral weight corresponds to an irreducible unitary representation with this weight

as its highest weight. For $U(M,N)$ on the other hand, there are two new phenomena. Firstly, there are further conditions on the dominant integral weight (m,n), namely that all entries of m should be different, and that all entries of n should be different. Secondly, the representation corresponding to (m,n), albeit a highest weight representation, does not in fact have (m,n) as its highest weight. Rather, the highest weight corresponding to (m,n) is 'renormalized': with $m_1 > m_2 > \ldots > m_M > 0$ and $n_1 > n_2 > \ldots > n_N > 0$, the highest weight (naively expected to be $(m_1,\ldots,m_M,-n_N,\ldots,-n_1)$) is in fact

$$(m_1 + \tfrac{1}{2}(N-M) + \tfrac{1}{2}, \ldots, m_i + \tfrac{1}{2}(N-M) + i - \tfrac{1}{2}, \ldots, m_M + \tfrac{1}{2}(N+M) - \tfrac{1}{2},$$

$$-(n_N + \tfrac{1}{2}(M+N) - \tfrac{1}{2}), \ldots, -(n_j + \tfrac{1}{2}(M-N) + j - \tfrac{1}{2}), \ldots, -(n_1 + \tfrac{1}{2}(M-N) + \tfrac{1}{2})).$$

Note that this highest weight is still dominant; however, it may no longer be integral, so that it defines a projective representation of $U(M,N)$ (single-valued on its double cover $\tilde{U}(M,N)$). These highest weight representations belong to the holomorphic discrete series of $U(M,N)$ [28].

The second problem is the quantization of $S = \mathcal{H} \otimes \mathbb{C}^{M+N}$, with the corresponding actions of $G = U_0(\mathcal{H})$ and $H = U(M,N)$. One regards $U(M,N)$ as a subgroup of $Sp(2(M+N),\mathbb{R})$, so that the symplectic action of the former on $\mathbb{C}^{M+N}$ is the restriction of the action of the latter [50, 24]. Due to the special way we defined the $U(M,N)$ action in subsection 3.2 as the inverse of a right-action, the quantization of this action of $Sp(2(M+N),\mathbb{R})$ is then given by the conjugate of the metaplectic representation π_m on $L^2(\mathbb{R}^{M+N}) \equiv \mathcal{L}$, cf. [25, 51, 50]. This defines a representation of the inverse image $\tilde{U}(M,N)$ of $U(M,N)$ in the metaplectic group $Mp(2(M+N),\mathbb{R})$ on $\overline{\mathcal{L}}$, which descends to a projective representation of $U(M,N)$, which we denote by $\pi_{\rm hf}(\tilde{U}(M,N))$. As pointed out in [51] and [6] (for $k=1$), this representation is precisely the one obtained from geometric quantization (in a suitable cohomological variant) if half-forms are taken into account. This yields a first candidate for the quantization of the $U(M,N)$ action on $\mathbb{C}^{M+N}$.

The second possibility is to take the tensor product of the (restriction of) the metaplectic representation of $\tilde{U}(M,N)$ with the square-root of the determinant,

which does define a unitary representation $\pi_{\rm sq}$ of $U(M,N)$ [51]; see [6] for a construction of this representation from geometric quantization. It is the representation which might be thought of as being defined by the physicists' second quantization on $\exp(\mathbb{C}^{M+N})$, as in the $U(M)$ case. However, since the action of $U(M,N)$ on $\mathbb{C}^{M+N}$ is not unitary, this second quantization is not, in fact, defined. In geometric quantization this lack of unitarity shows up through the non-existence of a totally complex invariant polarization on S which is positive. Consequently, one needs to work with an indefinite such polarization [6], and this leads to complications that will eventually cause a shift in the representations one would naively expect to occur in the decomposition of the quantization of S.

For finite-dimensional $\mathcal{H} = \mathbb{C}^k$ we therefore have a suitable quantization of $S = \mathbb{C}^k \otimes \mathbb{C}^{M+N}$, namely the Hilbert space $\overline{\mathcal{L}}_k \equiv \otimes^k \overline{\mathcal{L}}$ (the Fock space realization of this space is not useful, so we drop the notation $\mathcal{F}$). Moreover, we have natural unitary representations $\otimes^k \pi_{\rm sq/hf}$ of $\tilde{U}(M,N)$ on $\overline{\mathcal{L}}_k$, which are quantizations of the symplectic action of $U(M,N)$ on S. Following our notation for $U(M)$, we refer to these representations as $\pi^{-1}_{R,{\rm sq/hf}}$.

In addition, the quantization of the $U(k)$ action on S may be found (much more easily) from geometric quantization with or without half-forms. The latter case, in which we call the representation $\pi_{L,{\rm sq}}(U(k))$, is explicitly given in [25]. Its half-form variant $\pi_{L,{\rm hf}}(U(k))$ differs from it by the determinant representation raised to the power $(M-N)/2$.

It follows from the theory of Howe dual pairs [21] that $\overline{\mathcal{L}}_k$ decomposes discretely under these representations. Starting with $\pi_{L,{\rm sq}}(U(k)) \otimes \pi^{-1}_{R,{\rm sq}}(U(M,N))$, the explicit decomposition of $\overline{\mathcal{L}}_k$ is given in [25] as (remember that we have to take the conjugate of the $U(M,N)$ modules, but not of the $U(k)$ modules used in [25], since our $U(k)$ action is the usual one; also, we use the conventions of [2] and [18] for labelling the highest weight, rather than those of [25] - this corresponds to an interchange of m and n)

$$\overline{\mathcal{L}}_k \overset{\rm sq}{\simeq} \bigoplus_{(\mathsf{m},\mathsf{n})} \mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \overline{\mathcal{H}}^{U(M,N)}_{(\mathsf{m}+k,\mathsf{n})}, \tag{3.17}$$

where the sum is over all pairs (m,n) as defined before, with zeros allowed, but

neither m nor n allowed to be empty. $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ as a representation space of $U(k)$ was defined in subsection 3.3, and $\mathcal{H}^{U(M,N)}_{(\mathsf{m}+k,\mathsf{n})}$ carries the unitary representation of $U(M,N)$ with highest weight (not subject to further 'renormalization')

$$(m_1+k,\ldots,m_i+k,\ldots,m_M+k,-n_N,\ldots,-n_j,\ldots,-n_1).$$

The decomposition under $\pi_{L,\mathrm{hf}}(U(k))\otimes\pi^{-1}_{R,\mathrm{hf}}(U(M,N))$, on the other hand, reads [18]

$$\overline{\mathcal{L}}_k \stackrel{\mathrm{hf}}{\simeq} \bigoplus_{(\mathsf{m},\mathsf{n})} \mathcal{H}^{\tilde{U}(k)}_{(\mathsf{m}+\frac{1}{2}(M-N),\mathsf{n}-\frac{1}{2}(M-N))} \otimes \overline{\mathcal{H}}^{\tilde{U}(M,N)}_{(\mathsf{m}+\frac{1}{2}k,\mathsf{n}+\frac{1}{2}k)}, \tag{3.18}$$

where the highest weight $(\mathsf{m}+\frac{1}{2}k,\mathsf{n}+\frac{1}{2}k)$ is explicitly given by

$$(m_1+k/2,\ldots,m_i+k/2,\ldots,m_M+k/2,-n_N-k/2,\ldots,n_j-k/2,\ldots,-n_1-k/2),$$

whereas $\mathcal{H}_{(\mathsf{m}+\frac{1}{2}(M-N),\mathsf{n}-\frac{1}{2}(M-N))}$ is the tensor product of $\mathcal{H}_{(\mathsf{m},\mathsf{n})}$, and $\mathbb{C}$, carrying the determinant representation of $U(k)$ to the power $(M-N)/2$, cf. [18]).

Working with (3.17 for the sake of concreteness, we now wish to apply Rieffel induction from a suitable representation of $H=U(M,N)$ to $\overline{\mathcal{L}}_k$ in order to extract the copy of $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ for the value of (m,n) selected by the representation we induce from. Firstly, we need a dense subspace $L\subset\overline{\mathcal{L}}_k$ such that the function $x\rightarrow(\pi^{-1}_{R,\mathrm{sq}}(x)\psi,\varphi)$ is in $L^1(H)$ for all $\psi,\varphi\in L$, cf. subsection 1.4. This is easily found: using the decomposition (3.17), we take L to consist of vectors having a finite number of components in the decomposition, each component of which is in the tensor product of $\mathcal{H}^{U(k)}_{\ldots}$ and the dense subspace of K-finite vectors in the other factor. Since each function of the type $x\rightarrow(\pi(x)\psi,\varphi)$, where π is in the discrete series, and ψ and φ are K-finite vectors, is in Harish-Chandra's Schwartz space [28] (which is a subspace of $L^1(H)$), this choice indeed satisfies the demand. (Based on the explicit realization of $\overline{\mathcal{L}}_k$ as a function space [25], a more 'geometric' choice of L may also be found.)

As we are going to induce from holomorphic discrete series representations of $U(M,N)$, let us examine the tensor product $\overline{\mathcal{H}}^{U(M,N)}_{(\mathsf{m}_1,\mathsf{n}_1)}\otimes\mathcal{H}^{U(M,N)}_{(\mathsf{m}_2,\mathsf{n}_2)}$. Recall that (m,n) (which here refers to the actual highest weight, rather than the dominant integral

weight that is subject to renormalization, as sketched above) defines a unitary irreducible representation $\pi_{(\mathsf{m},\mathsf{n})}$ of the maximal compact subgroup $K = U(M) \times U(N)$ with highest weight $(m_1, \ldots, m_M, -n_N, \ldots, -n_1)$. By Theorem 2 in [46], the above tensor product is unitarily equivalent as a representation space of $U(M,N)$ to the representation induced (in the usual, Mackey, sense) from $\overline{\pi}_{(\mathsf{m}_1,\mathsf{n}_1)} \otimes \pi_{(\mathsf{m}_2,\mathsf{n}_2)}(K)$. Using the reduction-induction theorem, we can therefore decompose this induced representation as a direct sum over the representations induced from the components in the decomposition of $\overline{\pi}_{(\mathsf{m}_1,\mathsf{n}_1)} \otimes \pi_{(\mathsf{m}_2,\mathsf{n}_2)}(K)$.

Let us examine a generic representation $\pi^\kappa(H)$ (realized on the Hilbert space $\mathcal{H}^\kappa$ of functions $\psi : G \to \mathcal{H}_\kappa$ satisfying the equivariance condition $\psi(xk) = \pi_\kappa(k^{-1})\psi(x)$) induced from an irreducible representation $\pi_\kappa(K)$. The Rieffel induction procedure produces the semi-definite form $(\cdot,\cdot)_0$ on $L \otimes \mathcal{H}_\chi$ (where, in this case, $\mathcal{H}_\chi = \mathcal{H}^{U(M,N)}_{(\mathsf{m},\mathsf{n})}$ for certain (m,n)). Using (3.17) and the previous paragraph, we find that $L \otimes \mathcal{H}_\chi$ is a certain dense subspace of a direct sum with components of the type $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \mathcal{H}^\kappa$, in which H acts trivially on the first factor. By our construction of L, each element of $L \otimes \mathcal{H}_\chi$ only has components in a finite number of these Hilbert spaces, so that we can investigate each component separately. (Had the number of components of elements of L been infinite, the study of $(\cdot,\cdot)_0$ would have been more involved, as this is an unbounded and non-closable quadratic form, so that $(\sum_i \psi_i, \varphi)_0 \neq \sum_i (\psi_i, \varphi)_0$ for infinite sums.)

Factorizing $\int_H dx = \int_N dn \int_K dk$ [28], it follows from the equivariance condition and the orthogonality relations for compact groups that in a given component $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \mathcal{H}^\kappa$ the expression $(\psi,\varphi)_0 = \int_H dx\, (\mathbb{I} \otimes \pi^\kappa(x)\psi, \varphi)$ vanishes unless π_κ is the identity representation π_{id} of K. Given a highest weight representation $\pi_\chi(H)$ we Rieffel-induce from, there exists a unique pair (m,n) for which $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \mathcal{H}^{\mathrm{id}}$ occurs in the decomposition of $\overline{\mathcal{L}}_k \otimes \mathcal{H}_\chi$ as a sum over induced representations of H in the above sense.

Let L^{id} be the projection of $L \otimes \mathcal{H}_\chi$ onto this $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})} \otimes \mathcal{H}^{\mathrm{id}}$. We define $\tilde{V} : L^{\mathrm{id}} \to \mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ by linear extension of $\tilde{V}\psi_1 \otimes \psi_2 = \psi_1 \int_H dx\, \psi_2(x)$ (where $\psi_1 \in \mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ and $\psi_2 \in \mathcal{H}^{\mathrm{id}} \subset L^2(G)$). The integral exists by our assumptions on L; moreover, the

explicit form of the inner product in $\mathcal{H}^{\mathrm{id}}$ (namely $(f,g) = \int_H dx\, f(x)\overline{g(x)}$, as K is compact) leads to the equality $(\tilde{V}\psi, \tilde{V}\varphi) = (\psi,\varphi)_0$ (where the inner product on the left-hand side is the one in $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$). We now extend $\tilde{V}$ to a map V from $L \otimes \mathcal{H}_\chi$ to $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$ by putting it equal to zero on all spaces involving a factor $\mathcal{H}^\kappa$, where $\kappa \neq \mathrm{id}$ (and equal to V on L^{id}, of course). Clearly, by this and the preceding paragraph,

$$(V\psi, V\varphi) = (\psi,\varphi)_0. \tag{3.19}$$

We are now in a standard situation in the theory of Riefel induction, in which we can identify the null space of $(\cdot,\cdot)_0$ with the kernel of V, and the induced space $\mathcal{H}^\chi$ (which, we recall, is the completion of the quotient of $L \otimes \mathcal{H}_\chi$ by this null space in the inner product obtained from this form) with the closure of the image of V. It is clear from our definition of L that the image of V actually coincides with $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$. Also, the definition of the induced representation π^χ of $G = U(k)$ on $\mathcal{H}^\chi$ immediately implies that $\pi^\chi \simeq \pi_{(\mathsf{m},\mathsf{n})}$. Finally, note that (3.19) shows explicitly that $(\cdot,\cdot)_0$ is positive semi-definite, a fact which was already certified by Prop. 2 in [30].

Putting these arguments together, we have proved:

Theorem 3 *Let $U(k)$ and $U(M,N)$ act on $S = \mathbb{C}^k \otimes \mathbb{C}^{M+N}$ (equipped with the symplectic form (3.4)) from the left and the right, respectively, in the natural way, and let $\overline{\mathcal{L}}_k$ be the quantization of S, with commuting representations of $U(k)$ and $U(M,N)$ on $\overline{\mathcal{L}}_k$ (which quantize the above symplectic actions) as given (up to conjugation of the representation of $U(M,N)$) by Kashiwara-Vergne [25].*

Then Rieffel induction on $\overline{\mathcal{L}}_k$ from the holomorphic discrete series representation of $U(M,N)$ with highest weight $(\mathsf{m}+k,\mathsf{n})$ (that is, the highest weight with components $(m_1+k,\ldots,m_M+k,-n_N,\ldots,-n_1)$) leads to an induced space $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$, which as a Rieffel-induced $U(k)$ module carries the representation $\pi_{(\mathsf{m},\mathsf{n})}(U(k))$ (which is the Young product of the representation with Young diagram m and the conjugate of the representation with Young diagram n).

Moreover, the induced space is empty if one induces from a highest weight representation of $U(M,N)$ of the form (m,n) in which at least one m_i is smaller than k, or is not integral.

3.5 Discussion

The last part of the theorem is particularly unpleasant for the quantization theory of constrained system, for it shows that Theorem 1 cannot really be 'quantized' unless m or n are empty. For we would naturally induce from the holomorphic discrete series representation of $U(M,N)$ having the 'renormalized' highest weight corresponding to a coadjoint orbit characterized by (m,n), as explained at the beginning of this subsection. But then for k large enough the induced space will be empty, rather than consisting of $\mathcal{H}^{U(k)}_{(\mathsf{m},\mathsf{n})}$, as desired. As we have seen, the induction procedure is only successful if we induce from a representation with highest weight $(\mathsf{m}+k,\mathsf{n})$, rather than from the ($k$-independent) renormalized weight we ought to use by first principles. This is bizarre, given that the original weight (m,n) (or the orbit it corresponds to) knows nothing about k or $U(k)$. In addition, even without this problem the induced space will often be empty, because the 'correct' renormalized highest weight one induces from may simply not occur in the Kashiwara-Vergne decomposition (3.17) because of the half-integral nature of its entries (which is a pure 'quantum' phenomenon). (In a rather different setting, the discrepancy for large k between the 'decomposition' of S into pairs of matched coadjoint orbits for $U(k)$ and $U(M,N)$, and the decomposition of $\overline{\mathcal{L}}_k$ under these groups, must have been noticed by Adams [3], who points out that there is a good correspondence for $k \leq \min{(M,N)}$ only.)

It is peculiar to the non-compact ($N \neq 0$) case that this difficulty even arises if the half-form correction to quantization is not applied. For (3.17) is the non-compact analogue of (3.14), and in the latter quantization clearly does commute with reduction. If we do incorporate half-forms, we obtain (3.18) for $U(M,N)$ and (3.16) for $U(M)$. In both cases the Rieffel induction process generically (that is, if $M \neq N$) fails to produce the correct representation of $U(k)$, even if one induces from a representation whose highest weight is renormalized (compared to the weight expected from the orbit correspondence) by the term $k/2$.

Finally, the passage from $\mathbb{C}^k$ to infinite-dimensional Hilbert spaces is tortuous whenever half-forms are used (the corrections being infinite for $k=\infty$), and in the

non-compact case even without these. This is partly because of the k-dependence of the highest weights of $U(M,N)$, and partly because $\mathcal{L}$ does not contain the identity representation of $U(M,N)$ (recall that in the compact case we used the carrier space $\mathbb{C}\Omega$ of this representation as the fixed 'tail' vector to construct the von Neumann infinite tensor product from).

Clearly, this situation deserves further study. We do not think it is an artifact of our proposal of using Rieffel induction in the quantization of constrained systems. In fact, this technique comprises the only method known to us which is precise enough to bring the embarrassment to light.

References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd ed. (Addison Wesley, Redwood City, 1985).

[2] J. Adams, "Unitary highest weight modules", Adv. Math. **63**, 113-137 (1987).

[3] J. Adams, "Coadjoint orbits and reductive dual pairs", Adv. Math. **63**, 138-151 (1987).

[4] E.M. Alfsen and F.W. Shultz, "Non-commutative spectral theory for affine function spaces on convex sets", Mem. Amer. Math. Soc. **172** (1976).

[5] R.J. Baston and M.G. Eastwood, *The Penrose transform: its interaction with representation theory* (Oxford University Press, Oxford, 1989).

[6] R.J. Blattner and J.H. Rawnsley, "Quantization of the action of $U(k,l)$ on $\mathbb{R}^{2(k+l)}$", J. Funct. Anal. **50**, 188-214 (1983).

[7] P. Bona, *Classical projections and macroscopic limits of quantum mechanical systems*, manuscript (1986).

[8] R.P. Boyer, "Representation theory of the Hilbert-Lie group $U(H)_2$", Duke Math. J. **47**, 325-344 (1980).

[9] R.P. Boyer, "Representation theory of infinite dimensional unitary groups", in *Representation theory of groups and algebras*, Contemp. Math. **145**, 381-392 (1993).

[10] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Deillard-Bleick, *Analysis, Manifolds, and Physics*, 2nd ed. (North-Holland, Amsterdam, 1982).

[11] R. Cirelli, A. Manià, and L. Pizzocchero, "Quantum mechanics as an infinite-dimensional Hamiltonian system with uncertainty structure: Part I", J. Math. Phys. **31**, 2891-2897 (1990).

[12] T. Enright, R. Howe, and N. Wallach, "A classification of highest weight modules", in *Representation theory of reductive groups*, ed. P.C. Trombi (Birkhäuser, Boston, 1983) pp. 97-143.

[13] J.M.G. Fell and R.S. Doran, *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles*, Volume 2 (Academic Press, Boston, 1988).

[14] G.B. Folland, *Harmonic analysis on phase space* (Princeton University Press, Princeton, 1989).

[15] M.J. Gotay, J.M. Nester, and G. Hinds, "Presymplectic manifolds and the Dirac-Bergmann theory of constraints", J. Math. Phys. **19**, 2388-2399 (1978).

[16] P.A. Griffiths and J. Harris, *Principles of algebraic geometry* (Wiley, New York, 1978)

[17] V. Guillemin and S. Sternberg, *Symplectic techniques in physics* (Cambridge University Press, Cambridge, 1984).

[18] R. Howe, "Reciprocity laws in the theory of dual pairs", in *Representation theory of reductive groups*, ed. P.C. Trombi (Birkhäuser, Boston, 1983) pp. 159-175.

[19] R. Howe, "Dual pairs in physics: harmonic oscillators, photons, electrons, and singletons", Lect. Appl. Math. **21**, 179-207 (1985).

[20] R. Howe, "*The classical groups* and invariants of binary forms", in *The mathematical heritage of Hermann Weyl*, Proc. Symp. Pure Math. **48**, ed. R.O. Wells (American Mathematical Society, Providence, 1988) pp. 133-166.

[21] R. Howe, "Remarks on classical invariant theory", Trans. Amer. Math. Soc. **313**, 539-570 (1989) (Erratum: *ibid.* **318**, 823 (1990)).

[22] H.P. Jakobsen, "On singular holomorphic representations", Inv. Math. **62**, 67-78 (1980).

[23] R.V. Kadison, "A representation theory for commutative topological algebras", Mem. Amer. Math. Soc. **7** (1951).

[24] D. Kazhdan, B. Kostant, and S. Sternberg, "Hamiltonian group actions and dynamical systems of Calogero type", Commun.Pure Appl.Math. **31**, 481-507 (1978).

[25] M. Kashiwara and M. Vergne, "On the Segal-Shale-Weil representations and harmonic polynomials", Inv. Math. **44**, 1-47 (1978).

[26] A.A. Kirillov, "Representations of the infinite dimensional unitary group", Soviet Math. Dokl. **14**, 1355-1358 (1973) (Russian original: Dokl. Akad. Nauk SSSR **212**, 288-290 (1973)).

[27] A.A. Kirillov, "The orbit method, II: Infinite-dimensional Lie groups and Lie algebras", in *Representation theory of groups and algebras*, Contemp. Math. **145**, 33-64 (1993).

[28] A.W. Knapp, *Representation theory of semi-simple groups* (Princeton University Press, Princeton, 1986).

[29] N.P. Landsman, "Classical and quantum representation theory", in *Proc. Seminarium mathematische structuren in de de veldentheorie*, ed. H. Pijls (CWI, Amsterdam, 1994, to appear).

[30] N.P. Landsman, "Rieffel induction as generalized quantum Marsden-Weinstein reduction", to appear in J. Geom. Phys. (1994).

[31] N.P. Landsman and U.A. Wiedemann, "Massless particles, electromagnetism, and Rieffel induction", to appear in Rev. Math. Phys. (1995).

[32] M. Lazard and J. Tits, "Domaines d'injectivité de l'application exponentielle", Topology **4**, 315-322 (1965/6).

[33] E. Lerman, R. Montgomery, and R. Sjamaar, "Examples of singular reduction", in *Symplectic geometry*, ed. D. Salamon, LMS Lecture Notes **192** (Cambridge University Press, Cambridge, 1993) pp. 127-155.

[34] J.E. Marsden, *Applications of Global Analysis in Mathematical Physics* (Publish or Perish, Boston, 1974).

[35] J.E. Marsden, *Lectures on mechanics*, LMS Lecture Notes **174** (Cambridge University Press, Cambridge, 1992).

[36] J.E. Marsden and A. Weinstein, "Reduction of symplectic manifolds with symmetries", Rep. Math. Phys. **5**, 121-130 (1974).

[37] K. Mikami and A. Weinstein, "Moments and reduction for symplectic groupoids", Publ.RIMS Kyoto Univ. **24**, 121-140 (1988).

[38] R. Montgomery, "Heisenberg and isoholonomic inequalities", in *Symplectic geometry and mathematical physics*, eds. P. Donato et.al. (Birkhäuser, Boston, 1991) pp. 303-325.

[39] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3d ed. (Springer, New York, 1993).

[40] G.I. Ol'shanskii, "Unitary representations of infinite-dimensional classical groups $U(p,\infty)$, $SO_0(p,\infty)$, $Sp(p,\infty)$, and the corresponding motion groups", Funct. Anal. Appl. **12**, 185-195 (1978) (Russian original: Funk. Anal. Pril. **12**, 32-44 (1978)).

[41] G.I. Ol'shanskii, "Description of unitary representations with highest weight for groups $U(p,q)$", Funct. Anal. Appl. **14**, 190-200 (1981) (Russian original: Funk. Anal. Pril. **14**, 32-44 (1980)).

[42] G.I. Ol'shanskii, "Construction of unitary representations of infinite-dimensional classical groups", Soviet Math. Dokl. **21**, 66-70 (1980) (Russian original: Dokl.Akad. Nauk SSSR **250**, 284-288 (1980)).

[43] G.I. Ol'shanskii, "Unitary representations of infinite-dimensional pairs (G,K) and the formalism of R. Howe", in *Representation of Lie groups and related topics*, eds. A.M. Vershik and D.P. Zhelobenko (Gordon and Breach, New York, 1990) pp. 269-464.

[44] G.I. Ol'shanskii, "Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians", Adv. Sov. Math. **2**, 1-66.

[45] M. Reed, and B. Simon, *Functional analysis* (Academic Press, New York, 1972).

[46] J. Repka, "Tensor products of holomorphic discrete series representations" Can. J. Math. **31**, 836-844 (1979).

[47] M.A. Rieffel, "Induced representations of C^*-algebras", Adv.Math. **13**, 176-257 (1974).

[48] M.A. Robson, "Geometric quantization of reduced cotangent bundles", to appear in J. Geom. Phys. (1995).

[49] F.W. Shultz, "Pure states as dual objects for C^*-algebras", Commun. Math. Phys. **82**, 497-509 (1982).

[50] S. Sternberg, "Some recent results on the metaplectic representation", in *Group theoretical methods in physics*, ed. P. Kramer and A. Rieckers, SLNP **79** (Springer, New York, 1978) pp. 117-143.

[51] S. Sternberg and J.A. Wolf, "Hermitian Lie algebras and metaplectic representations", Trans. Amer. Math. Soc. **238**, 1-43 (1978).

[52] G.M. Tuynman, *Studies in geometric quantization*, PhD thesis, University of Amsterdam (1988).

[53] G.M. Tuynman and W.A.J.J. Wiegerinck, "Central extensions in physics", J. Geom. Phys. **4**, 207-258 (1987).

[54] D.A. Vogan, "Unitary representations of reductive Lie groups and the orbit method", in *New developments in Lie theory*, ed. J. Tirao and N. Wallach (Birkhäuser, Boston, 1992) pp. 87-114.

[55] J. von Neumann, "On infinite direct products", Compos. Math. **6**, 1-77 (1938).

[56] N.E. Wegge-Olsen, *K-theory and C^*-algebras* (Oxford University Press, Oxford, 1993).

[57] A. Weinstein, "The local structure of Poisson manifolds", J. Diff. Geom. **18**, 523-557 (1983).

[58] H. Weyl, *The classical groups* (Princeton University Press, Princeton, 1946).

[59] N. Woodhouse, *Geometric quantization*, 2nd ed. (Oxford University Press, Oxford, 1992).

[60] P. Xu, "Morita equivalence of Poisson manifolds", Commun. Math. Phys. **142**, 493-509 (1991).

[61] D.P. Zhelobenko, *Compact Lie groups and their representations*, Transl. Math. Mon. **40** (Amer. Math. Soc., Providence, 1973).